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Transitivity of parametric family of cardinality-based fuzzy similarity measures using Lukasiewicz t-norm

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Abstract

In fuzzy logic, where members of a set might be linguistic terms, the degree of reflexivity might be in unit interval $[0, 1]$ instead of $\{0, 1\}$. This behaviour of a fuzzy set plays an important role especially in the field of inclusion and similarity measure. This paper is aimed at discovering the relations between the parameters of Łukasiewicz transitive members of a family of cardinality-based fuzzy measure.

Keywords: Cardinality based similarity measures, T-transitivity; Łuckaseiwicz T-norm.

1 Introduction

In everyday life, one is often faced situations in which one needs to distinguish between multiple objects. This process is usually carried out by means

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of measures based on the common features observed in the objects or the features of one object contained by the other and commonly known as similarity and inclusion measures. The origin of many similarity measures can be seen in Taxonomy, where feature-based approach is followed. When studying the Mathematics of similarity, the objects are normally represented by the sets of features. For example, when comparing two cars these sets may contain their colors, safety ratings and their performances. Therefore, similarity and inclusion measures are very useful techniques to organize or classify objects, form concepts and to make generalizations. It is a very important concept in many scientific fields such as Bioinformatics, Biology, Chemistry, Information retrieval, Statistics and many others [1], [2], [3], [4], [5] and [6].

Similarity measure determines the degree to which the two objects resembles to each other while the inclusion measure expresses the degree to which the characteristics of the one object include another. The degree of similarity and inclusion measure has received much more attention in recent decades because these are important tools for decision making, pattern recognition, medical diagnosis, and data mining applications.

Fuzzy sets introduced by Zadeh [7] in 1965 have an edge over the crisp sets that they can represent the degrees of truth or falsehood. Consequently, in many practical applications, fuzzy similarity measures which represent the degree of similarity or the measure of how much similar two or more elements are seem much closer to reality than their crisp counterparts do. Since their inception in 1973, fuzzy similarity and inclusion measures and their place in almost all application areas from A to Z (i.e., from Anthropology to Zoology). Cardinality based similarity and inclusion measures play a key role in literature and applications [8], [9], [10] and [11]. Some memorable contributions can be found in [12], [13], [14], [15], [16] and [17].

Cardinality based similarity and inclusion measures rely on the cardinalities of sets of common and different features. New measures were introduced by assigning different weights to the sets of common and different features. Such families of similarity and inclusion measures are known as parametric families and these weight are termed as parameters. Some famous parametric families were introduced by Tversky [5], Gower & Legendre [18] and De Baets et al [14]. A family of rational expressions using two, three and four parameters [19], [20], [21] and [16] are available in literature. While axioms for fuzzy similarity measures are focused, fuzzy equivalence relation stands as the best model for fuzzy similarity measure. A fuzzy equivalence relation is reflexive, symmetric and T-transitive [22] fuzzy relation. Unfortunately, the majority of fuzzy similarity measures found in literature do not follow one or the other desired form of fuzzy transitivity, in particular, the axiom of T-transitivity is violated. In [16] Janssens et al established conditions under which a parametric family of cardinality based fuzzy similarity measures become T-transitive. The significant development of meta-theorems, which ensure T-transitivity, can be found in [23] and [17]. These theorems are used to construct the necessary and sufficient conditions to obtain transitivity.

De Baets et al introduced a parametric family of cardinality based similarity measure [13] and inclusion measure [14], which cover most of the famous measures found in literature for different values of parameters. Basically, these measures used eight parameters and Janssens et al characterized the T-transitive members of the family of similarity and inclusion measures for only four parameters for Łukasiewicz, Product and Min t-norms.

Sometimes in similarity or inclusion measure, the features which are absent or negative match play a key role in the comparison of two objects and it seems improper to ignore these negative match features. In this context, a manuscript is already been submitted by Javed, Samina and Syed using six parameters to characterize the Łukasiewicz-transitive members, while this submission not only tried to explore the transitivity between the objects based on the similarities but also to incorporate the negative matching features. This paper is focused to determine the necessary conditions using all eight parameters for Łukasiewicz transitive family of similarity and inclusion measures based on cardinality. It is an effort to include some other measures in the parametric family of measure, which were put out of action by the restriction of four parameters.

The notion of fuzzy set was introduced by Zadeh in 1965 in his seminal paper [7]. A fuzzy set A is a mapping from a universe X to $[0,1]$. For any $x \in X$, the value $A(x)$ denotes the degree of membership of x in A. Let $F(X)$ be the set of all fuzzy subsets of a universe X. For a crisp universe X, a fuzzy subset of $X \times X$ is called a fuzzy binary relation and throughout this paper we termed fuzzy binary relations as fuzzy relations. Given a crisp universe X, and $A, B \in F(X)$, A is said to be a subset of B (in Zadeh's sense [7]) denoted by $A \subseteq B$, if and only if $A(x) \leq B(x)$ for all $x \in X$.

Definition 1.1 [24] The triangular norm (t-norm) T and triangular conorm (t-conorm) T^* are increasing, associative, commutative and mapping $[0,1]^2 \to [0,1]$ satisfying $T(1,x) = x$ and $T^*(x,0) = x$ for all $x \in [0,1]$.

To every t-norm T there corresponds a t-conorm T^* called the dual t-conorm, defined by: $T^*(x,y) = 1 - T(1-x, 1-y)$. For the Lukasiewicz t-norm $W(x,y) = \max(x+y-1,0)$, the corresponding t-conorm is $W^*(x,) = \min(x+y,1)$.

Definition 1.2 [25] A negator N is an order-reversing $[0,1] \to [0,1]$ mapping such that $N(0) = 1$ and $N(1) = 0$. A strictly decreasing negator satisfying $N(N(x)) = x$ for all $x \in [0,1]$ is called a strong negator.

The negator defined as: $N(x) = 1 - x$ for all $x \in X$, is called standard negator and was defined by Zadeh himself.

Definition 1.3 [22] Given a t-norm T, a $T-$equivalence relation on a set X is a fuzzy relation E on X that satisfies:

(i) $E(x,x) = 1$ for all $x \in X$; (Reflexivity),

(ii) $E(x,y) = E(y,x)$ for all $x, y \in X$; (Symmetry),

(iii) $T(E(x,y), E(y,z)) \leq E(x,z)$ for all $x, y, z \in X$; (T-transitivity).

Definition 1.4 [25] Let R be a fuzzy relation on X. A triangular norm T is T-transitive if and only if for all $x, y, z \in X$,

$$T\left(R\left(x,y\right), R\left(y,z\right)\right) \leq R\left(x,z\right)$$

Definition 1.5 [22] An inclusion measure for ordinary sets is binary fuzzy relation I on the power set $P\left(X\right) = \{0,1\}^X$ satisfying,

$$A \subset B \Rightarrow I\left(A,B\right) = 1$$

2 Transitivity of Inclusion Measures

The rational cardinality based inclusion measure was proposed by Janssens et al [14] in 2002. They define inclusion measure for two subsets A and B of a finite universe X as

$$I(A,B) = \frac{x\chi_{A,B} + t\chi_{B,A} + y\delta_{A,B} + z\nu_{A,B}}{x'\chi_{A,B} + t'\chi_{B,A} + y'\delta_{A,B} + z'\nu_{A,B}} \tag{1}$$

with $\chi_{A,B} = |A \backslash B|, \chi_{B,A} = |B \backslash A|$ and the parameters $x, x', t, t', y, y', z, z' \in \{0,1\}$. These parameters are considered to be positive real numbers. The conditions of $x < x'$ and $t < t'$ are imposed to contain inclusion measure $I(A,B)$ in unit interval, as $x = x', t = t'$ leads to the trivial case, so we consider $0 \leq x < x'$ and $0 \leq t < t'$.

Some inclusion measures $I_1 - I_5$ [14],[26] available in literature are member of this parametric family given in Table 1. The inclusion measure I_6 given by Kuncheva [27] also becomes a part of this parametric family by considering all

Tab. 1: Some members of Family (1).

Measure	*Expression*	x	x'	t	t'	y	y'	z	z'
I_1	$\frac{\lvert B\setminus A\rvert}{\lvert A\triangle B\rvert}$	0	1	1	1	0	0	0	0
I_2	$\frac{\lvert A^c\rvert}{\lvert(A\cap B)^c\rvert}$	0	1	1	1	0	0	1	1
I_3	$\frac{\lvert B\rvert}{\lvert A\cup B\rvert}$	0	1	1	1	1	1	0	0
I_4	$\frac{\lvert(A\setminus B)^c\rvert}{n}$	0	1	1	1	1	1	1	1
I_5	$1-\frac{2\lvert A\setminus B\rvert}{n+\lvert A\triangle B\rvert}$	0	2	2	2	1	1	1	1
I_6	$\frac{\lvert A\cap B\rvert}{\lvert A\rvert}$	0	1	0	0	1	1	0	0

eight parameters. This inclusion measure does not belong to the parametric family of four parameters.

Janssens et al [16] used only four parameters x, x', y and z to explore the relations between them for T -transitivity. In this section, we determine the conditions on all eight parameters x, x', t, t', y, y', z and z' for T -transitivity. For this purpose, the notion of fuzzy reflexivity of a measure is used. The inclusion measure will be fuzzy reflexive if $y' = \theta y,\ z' = \varphi z$ for $\theta, \varphi \geq 1$, so the above inclusion measure can be written as

$$I(A,B) = \frac{x\chi_{A,B} + t\chi_{B,A} + y\delta_{A,B} + z\nu_{A,B}}{x'\chi_{A,B} + t'\chi_{B,A} + y\theta\delta_{A,B} + z\varphi\nu_{A,B}} \tag{2}$$

with $\theta, \varphi \geq 1$. For six parameters, the above inclusion measure can be written as

$$I(A,B) = \frac{x\chi_{A,B} + x'\chi_{B,A} + y\delta_{A,B} + z\nu_{A,B}}{x'\ \triangle_{A,B} + y\theta\delta_{A,B} + z\varphi\nu_{A,B}} \tag{3}$$

where $y' = \theta y,\ z' = \varphi z$ for $\theta, \varphi \geq 1$. In this paper, Łukasiewicz t-norm is used to articulate the conditions for transitivity, which is termed as W-transitivity throughout the paper.

Theorem 1 The W-transitive members of the class of inclusion measures (3) are characterized by the necessary conditions

$$x' \geq \max(y\theta, z\varphi)$$

Proof: To determine the conditions on parameters x, x', y, and z for transitivity, the inclusion measure S is W-transitive if it holds that,

$$\begin{aligned} W(S(A,B), S(B,C)) &\leq S(A,C) \\ S(A,B) + S(B,C) - 1 &\leq S(A,C) \end{aligned}$$

Applying the negator $N(x) = 1 - x$, on both sides, we get

$$\begin{aligned} N\left(S(A,B)+S(B,C)-1\right) &\leq N\left(S(A,C)\right) \\ 1-\left(S(A,B)+S(B,C)-1\right) &\geq 1-S(A,C) \\ [1-S(A,B)]+[1-S(B,C)]-[1-S(A,C)] &\geq 0 \end{aligned}$$

In terms of family (3), the above inequality can be written as

$$\begin{aligned} &\left[1-\frac{x\chi_{A,B}+x'\chi_{B,A}+y\delta_{A,B}+z\nu_{A,B}}{x'\bigtriangleup_{A,B}+y\theta\delta_{A,B}+z\varphi\nu_{A,B}}\right] \\ &+\left[1-\frac{x\chi_{B,C}+x'\chi_{C,B}+y\delta_{B,C}+z\nu_{B,C}}{x'\bigtriangleup_{B,C}+y\theta\delta_{B,C}+z\varphi\nu_{B,C}}\right] \\ &-\left[1-\frac{x\chi_{A,C}+t\chi_{C,A}+y\delta_{A,C}+z\nu_{A,C}}{x'\bigtriangleup_{A,C}+y\theta\delta_{A,C}+z\varphi\nu_{A,C}}\right]\geq 0 \\ \Rightarrow &\left[\frac{(x'-x)\,\chi_{A,B}+y\,(\theta-1)\,\delta_{A,B}+z\,(\varphi-1)\,\nu_{A,B}}{x'\bigtriangleup_{A,B}+y\theta\delta_{A,B}+z\varphi\nu_{A,B}}\right] \\ &+\left[\frac{(x'-x)\,\chi_{B,C}+y\,(\theta-1)\,\delta_{B,C}+z\,(\varphi-1)\,\nu_{B,C}}{x'\bigtriangleup_{B,C}+y\theta\delta_{B,C}+z\varphi\nu_{B,C}}\right] \\ &-\left[\frac{(x'-x)\,\chi_{A,C}+y\,(\theta-1)\,\delta_{A,C}+z\,(\varphi-1)\,\nu_{A,C}}{x'\bigtriangleup_{A,C}+y\theta\delta_{A,C}+z\varphi\nu_{A,C}}\right]\geq 0 \end{aligned}$$

where $\chi_{A,B} = |A\backslash B|, \delta_{A,B} = |A\cap B|, \nu_{A,B} = |(A\cup B)^c|, \bigtriangleup_{A,B} = |A\backslash B| + |B\backslash A|$, etc. This implies that

$$\begin{aligned} &\left[\frac{(x'-x)\,|A\backslash B|+y\,(\theta-1)\,|A\cap B|+z\,(\varphi-1)\,|(A\cup B)^c|}{x'\,(|A\backslash B|+|B\backslash A|)+y\theta\,|A\cap B|+z\varphi\,|(A\cup B)^c|}\right] \\ &+\left[\frac{(x'-x)\,|B\backslash C|+y\,(\theta-1)\,|B\cap C|+z\,(\varphi-1)\,|(B\cup C)^c|}{x'\,(|B\backslash C|+|C\backslash B|)+y\theta\,|B\cap C|+z\varphi\,|(B\cup C)^c|}\right] \\ &-\left[\frac{(x'-x)\,|A\backslash C|+y\,(\theta-1)\,|A\cap C|+z\,(\varphi-1)\,|(A\cup C)^c|}{x'\,(|A\backslash C|+|C\backslash A|)+y\theta\,|A\cap C|+z\varphi\,|(A\cup C)^c|}\right]\geq 0 \qquad (4) \end{aligned}$$

Consider the setting in figure 1, then following conditions hold,

$$\begin{array}{ll} |A\backslash B| = a_1 + b_2 & |B\backslash A| = a_2 + b_1 \\ |B\backslash C| = a_2 + b_3 & |C\backslash B| = a_3 + b_2 \\ |A\backslash C| = a_1 + b_3 & |C\backslash A| = a_3 + b_1 \\ |A\cap B| = b_3 + c & |(A\cup B)^c| = a_3 + d \\ |A\cap C| = b_2 + c & |(A\cup C)^c| = a_2 + d \\ |B\cap C| = b_1 + c & |(B\cup C)^c| = a_1 + d \end{array} \quad \begin{array}{l} |A \triangle B| = a_1 + b_2 + a_2 + b_1 \\ |A \triangle C| = a_1 + b_3 + a_3 + b_1 \\ |B \triangle C| = a_2 + b_3 + a_3 + b_2 \end{array} \tag{5}$$

where $a_{i's}, b_{i's}, c$ and d are cardinalities.

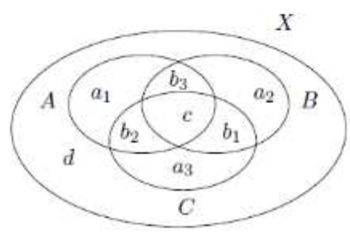

Fig. 1: Cardinalities associated with sets A, B and C.

Using the values of cardinalities given in (5), the inequality (4) implies

$$\left[\frac{(x'-x)(a_1+b_2)+y(\theta-1)(b_3+c)+z(\varphi-1)(a_3+d)}{x'(a_1+b_2+a_2+b_1)+y\theta(b_3+c)+z\varphi(a_3+d)}\right]$$
$$+\left[\frac{(x'-x)(a_2+b_3)+y(\theta-1)(b_1+c)+z(\varphi-1)(a_1+d)}{x'(a_2+b_3+a_3+b_2)+y\theta(b_1+c)+z\varphi(a_1+d)}\right]$$
$$-\left[\frac{(x'-x)(a_1+b_3)+y(\theta-1)(b_2+c)+z(\varphi-1)(a_2+d)}{x'(a_1+b_3+a_3+b_1)+y\theta(b_2+c)+z\varphi(a_2+d)}\right] \geq 0 \tag{6}$$

Setting $a_1 = b_1 = c = d = 0$, (6) implies

$$\left[\frac{(x'-x)b_2+y(\theta-1)b_3+z(\varphi-1)a_3}{x'(b_2+a_2)+y\theta b_3+z\varphi a_3}\right]+\left[\frac{(x'-x)(a_2+b_3)}{x'(a_2+b_3+a_3+b_2)}\right]$$
$$-\left[\frac{(x'-x)b_3+y(\theta-1)b_2+z(\varphi-1)a_2}{x'(b_3+a_3)+y\theta b_2+z\varphi a_2}\right] \geq 0$$

$$\begin{aligned}
\Rightarrow\quad (x'-x) & \left[\begin{array}{c} \frac{b_2}{x'(b_2+a_2)+y\theta b_3+z\varphi a_3}+\frac{a_2+b_3}{x'(a_2+b_3+a_3+b_2)} \\ -\frac{b_3}{x'(b_3+a_3)+y\theta b_2+z\varphi a_2} \end{array}\right] \\
+y(\theta-1) & \left[\frac{b_3}{x'(b_2+a_2)+y\theta b_3+z\varphi a_3}-\frac{b_2}{x'(b_3+a_3)+y\theta b_2+z\varphi a_2}\right] \\
+z(\varphi-1) & \left[\frac{a_3}{x'(b_2+a_2)+y\theta b_3+z\varphi a_3}-\frac{a_2}{x'(b_3+a_3)+y\theta b_2+z\varphi a_2}\right] \geq 0
\end{aligned}$$

Since $x' > x, \theta > 1$ and $\varphi > 1$ so the factors $x'-x, \theta-1$ and $\varphi-1$ can be omitted and we obtain

$$\left[\begin{array}{c} \frac{b_2}{x'(b_2+a_2)+y\theta b_3+z\varphi a_3}+\frac{a_2+b_3}{x'(a_2+b_3+a_3+b_2)} \\ -\frac{b_3}{x'(b_3+a_3)+y\theta b_2+z\varphi a_2} \end{array}\right] \geq 0 \tag{7}$$

In particular, setting $a_2 = a_3 = 0$ in (7) leads to

$$\begin{aligned}
\frac{b_2}{x'b_2+y\theta b_3}+\frac{b_3}{x'(b_3+b_2)}-\frac{b_3}{x'b_3+y\theta b_2} &\geq 0 \\
[x'b_2+2x'b_3]\,y\theta b_2+\left[y^2\theta^2+(x')^2-x'y\theta\right]b_3^2 &\geq 0 \\
(x')^2 &\geq (x'-y\theta)\,y\theta \\
x' &\geq y\theta. \qquad (8)
\end{aligned}$$

Other combinations of $a_{i's}$ and $b_{i's}$, that is, $b_2 = b_3 = 0, a_2 = b_3 = 0$, or $b_2 = a_3 = 0$ do not lead to any other conditions on x, x', y, z. Setting $a_2 = b_2 = c = d = 0$, (6) implies

$$\begin{aligned}
&\left[\frac{(x'-x)\,a_1+y(\theta-1)\,b_3+z(\varphi-1)\,a_3}{x'(a_1+b_1)+y\theta b_3+z\varphi a_3}\right] \\
&+\left[\frac{(x'-x)\,b_3+y(\theta-1)\,b_1+z(\varphi-1)\,a_1}{x'(b_3+a_3)+y\theta b_1+z\varphi a_1}\right] \\
&-\left[\frac{(x'-x)(a_1+b_3)}{x'(a_1+b_3+a_3+b_1)}\right] \geq 0
\end{aligned}$$

$$\begin{aligned}
\Rightarrow\quad (x'-x) & \left[\begin{array}{c} \frac{a_1}{x'(a_1+b_1)+y\theta b_3+z\varphi a_3}+\frac{b_3}{x'(b_3+a_3)+y\theta b_1+z\varphi a_1} \\ -\frac{a_1+b_3}{x'(a_1+b_3+a_3+b_1)} \end{array}\right] \\
+y(\theta-1) & \left[\frac{b_3}{x'(a_1+b_1)+y\theta b_3+z\varphi a_3}+\frac{b_1}{x'(b_3+a_3)+y\theta b_1+z\varphi a_1}\right]
\end{aligned}$$

$$+z\left(\varphi-1\right)\left[\frac{a_3}{x'\left(a_1+b_1\right)+y\theta b_3+z\varphi a_3}+\frac{a_1}{x'\left(b_3+a_3\right)+y\theta b_1+z\varphi a_1}\right]\geq 0.$$

Since $x'>x,\theta>1$ and $\varphi>1$ so the factors $x'-x,\theta-1$ and $\varphi-1$ can be omitted and we obtain

$$\left[\begin{array}{c}\frac{a_1}{x'(a_1+b_1)+y\theta b_3+z\varphi a_3}+\frac{b_3}{x'(b_3+a_3)+y\theta b_1+z\varphi a_1}\\-\frac{a_1+b_3}{x'(a_1+b_3+a_3+b_1)}\end{array}\right]\geq 0 \tag{9}$$

In particular, setting $a_1=a_3=0$ in inequality (9) leads to the same result obtained in (8). Similarly, setting of $b_1=b_3=0$ in inequality (9) implies

$$\begin{aligned}\frac{a_1}{x'a_1+z\varphi a_3}-\frac{a_1}{x'\left(a_1+a_3\right)} &\geq 0\\ x' &\geq z\varphi \qquad (10)\end{aligned}$$

and setting of $b_1=a_3=0$ in (9) ends up with

$$\begin{aligned}\frac{a_1}{x'a_1+y\theta b_3}+\frac{b_3}{x'b_3+z\varphi a_1}-\frac{1}{x'} &\geq 0\\ \left(x'\right)^2 &\geq yz\theta\varphi \qquad (11)\end{aligned}$$

Other combinations of $a_{i's}$ and $b_{i's}$ do not lead to any other conditions on x,x',y,z. Setting $a_3=b_3=c=d=0$ in (6) implies

$$\begin{aligned}&\left[\frac{\left(x'-x\right)\left(a_1+b_2\right)}{x'\left(a_1+b_2+a_2+b_1\right)}\right]\\&+\left[\frac{\left(x'-x\right)a_2+y\left(\theta-1\right)b_1+z\left(\varphi-1\right)a_1}{x'\left(a_2+b_2\right)+y\theta b_1+z\varphi a_1}\right]\\&-\left[\frac{\left(x'-x\right)a_1+y\left(\theta-1\right)b_2+z\left(\varphi-1\right)a_2}{x'\left(a_1+b_1\right)+y\theta b_2+z\varphi a_2}\right]\geq 0\end{aligned}$$

$$\begin{aligned}&\left(x'-x\right)\left[\begin{array}{c}\frac{a_1+b_2}{x'(a_1+b_2+a_2+b_1)}+\frac{a_2}{x'(a_2+b_2)+y\theta b_1+z\varphi a_1}\\-\frac{a_1}{x'(a_1+b_1)+y\theta b_2+z\varphi a_2}\end{array}\right]\\&+y\left(\theta-1\right)\left[\frac{b_1}{x'\left(a_2+b_2\right)+y\theta b_1+z\varphi a_1}-\frac{b_2}{x'\left(a_1+b_1\right)+y\theta b_2+z\varphi a_2}\right]\end{aligned}$$

$$+z\left(\varphi-1\right)\left[\frac{a_1}{x'\left(a_2+b_2\right)+y\theta b_1+z\varphi a_1}-\frac{a_2}{x'\left(a_1+b_1\right)+y\theta b_2+z\varphi a_2}\right]\geq 0. \tag{12}$$

Since $x'>x$, $\theta>1$ and $\varphi>1$ so the factors $x'-x, \theta-1$ and $\varphi-1$ can be omitted. In particular, setting $b_1=b_2=0$ in the inequality (12) leads to the results obtained in (10) and no other combinations lead to other conditions on x, x', y and z. Thus all outcomes (8 and 10) can be combined to

$$x'\geq \max\left(y\theta, z\varphi\right)$$

as the necessary condition for the family to be W-transitivity.

Corollary 1.1 If $\theta=\varphi$, then W-transitive members of the class of inclusion measure (3) are characterized by the necessary conditions

$$x'\geq \max(y\theta, z\theta)$$

Corollary 1.2 If $\theta=1=\varphi$, then W -transitive members of the class of inclusion measure (3) are characterized by the necessary conditions

$$x'\geq \max(y, z)$$

Theorem 2 The W-transitive members of the class of inclusion measures (2) are characterized by the necessary conditions

$$\begin{aligned}\min\left(\, x', t'\right) &\geq \max\left(y\theta, z\varphi\right)\\ \wedge(x')^2 &\geq \max(y\theta\left(t'-y\theta\right), z\varphi\left(t'-z\varphi\right))\\ \wedge(t')^2 &\geq \max(y\theta\left(x'-y\theta\right), z\varphi\left(x'-z\varphi\right))\end{aligned}$$

Proof: To determine the conditions on parameters x, x', y, and z for transitivity, Let us suppose that the inclusion measure S is W-transitive i.e.,

$$\begin{aligned}W\left(S(A,B), S(B,C)\right) &\leq S(A,C)\\ S(A,B)+S(B,C)-1 &\leq S(A,C)\end{aligned}$$

Applying the negator $N(x)=1-x$, on both sides, we get

$$[1-S(A,B)]+[1-S(B,C)]-[1-S(A,C)]\geq 0$$

In terms of family (2), the above inequality can be written as

$$\left[1-\frac{x\chi_{A,B}+t\chi_{B,A}+y\delta_{A,B}+z\nu_{A,B}}{x'\chi_{A,B}+t'\chi_{B,A}+y\theta\delta_{A,B}+z\varphi\nu_{A,B}}\right]$$

$$+\left[1-\frac{x\chi_{B,C}+t\chi_{C,B}+y\delta_{B,C}+z\nu_{B,C}}{x'\chi_{B,C}+t'\chi_{C,B}+y\theta\delta_{B,C}+z\varphi\nu_{B,C}}\right]$$

$$-\left[1-\frac{x\chi_{A,C}+t\chi_{C,A}+y\delta_{A,C}+z\nu_{A,C}}{x'\chi_{A,C}+t'\chi_{C,A}+y\theta\delta_{A,C}+z\varphi\nu_{A,C}}\right]\geq 0$$

$$\Rightarrow\left[\frac{(x'-x)\,\chi_{A,B}+(t'-t)\,\chi_{B,A}+y\,(\theta-1)\,\delta_{A,B}+z\,(\varphi-1)\,\nu_{A,B}}{x'\chi_{A,B}+t'\chi_{B,A}+y\theta\delta_{A,B}+z\varphi\nu_{A,B}}\right]$$

$$+\left[\frac{(x'-x)\,\chi_{B,C}+(t'-t)\,\chi_{C,B}+y\,(\theta-1)\,\delta_{B,C}+z\,(\varphi-1)\,\nu_{B,C}}{x'\chi_{B,C}+t'\chi_{C,B}+y\theta\delta_{B,C}+z\varphi\nu_{B,C}}\right]$$

$$-\left[\frac{(x'-x)\,\chi_{A,C}+(t'-t)\,\chi_{C,A}+y\,(\theta-1)\,\delta_{A,C}+z\,(\varphi-1)\,\nu_{A,C}}{x'\chi_{A,C}+t'\chi_{C,A}+y\theta\delta_{A,C}+z\varphi\nu_{A,C}}\right]\geq 0$$

where $\chi_{A,B}=|A\backslash B|\,,\chi_{B,A}=|B\backslash A|\,,\chi_{B,C}=|B\backslash C|\,,\chi_{C,B}=|C\backslash B|\,,\chi_{A,C}=|A\backslash C|\,,\chi_{C,B}=|C\backslash A|\,.$

$$\Rightarrow\left[\frac{(x'-x)\,|A\backslash B|+(t'-t)\,|B\backslash A|+y\,(\theta-1)\,|A\cap B|+z\,(\varphi-1)\,|(A\cup B)^c|}{x'\,|A\backslash B|+t'\,|B\backslash A|+y\theta\,|A\cap B|+z\varphi\,|(A\cup B)^c|}\right]$$

$$+\left[\frac{(x'-x)\,|B\backslash C|+(t'-t)\,|C\backslash B|+y\,(\theta-1)\,|B\cap C|+z\,(\varphi-1)\,|(B\cup C)^c|}{x'\,|B\backslash C|+t'\,|C\backslash B|+y\theta\,|B\cap C|+z\varphi\,|(B\cup C)^c|}\right]$$

$$-\left[\frac{(x'-x)\,|A\backslash C|+(t'-t)\,|C\backslash A|+y\,(\theta-1)\,|A\cap C|+z\,(\varphi-1)\,|(A\cup C)^c|}{x'\,|A\backslash C|+t'\,|C\backslash A|+y\theta\,|A\cap C|+z\varphi\,|(A\cup C)^c|}\right]\geq 0.$$

Considering the settings in figure and using the values of cardinalities given in (5), the above inequality leads to

$$\left[\frac{(x'-x)\,(a_1+b_2)+(t'-t)\,(a_2+b_1)+y\,(\theta-1)\,(b_3+c)+z\,(\varphi-1)\,(a_3+d)}{x'\,(a_1+b_2)+t'\,(a_2+b_1)+y\theta\,(b_3+c)+z\varphi\,(a_3+d)}\right]$$

$$+\left[\frac{(x'-x)\,(a_2+b_3)+(t'-t)\,(a_3+b_2)+y\,(\theta-1)\,(b_1+c)+z\,(\varphi-1)\,(a_1+d)}{x'\,(a_2+b_3)+t'\,(a_3+b_2)+y\theta\,(b_1+c)+z\varphi\,(a_1+d)}\right]$$

$$-\left[\frac{(x'-x)\,(a_1+b_3)+(t'-t)\,(a_3+b_1)+y\,(\theta-1)\,(b_2+c)+z\,(\varphi-1)\,(a_2+d)}{x'\,(a_1+b_3)+t'\,(a_3+b_1)+y\theta\,(b_2+c)+z\varphi\,(a_2+d)}\right]\geq 0.\qquad(13)$$

Setting $a_1 = b_1 = c = d = 0$, (13) implies

$$\left[\frac{(x'-x)\,b_2 + (t'-t)\,a_2 + y\,(\theta-1)\,b_3 + z\,(\varphi-1)\,a_3}{x'b_2 + t'a_2 + y\theta b_3 + z\varphi a_3}\right] + \left[\frac{(x'-x)\,(a_2+b_3) + (t'-t)\,(a_3+b_2)}{x'\,(a_2+b_3) + t'\,(a_3+b_2)}\right]$$

$$-\left[\frac{(x'-x)\,b_3 + (t'-t)\,a_3 + y\,(\theta-1)\,b_2 + z\,(\varphi-1)\,a_2}{x'b_3 + t'a_3 + y\theta b_2 + z\varphi a_2}\right] \geq 0$$

$$\begin{aligned} \Rightarrow\quad & (x'-x)\left[\begin{array}{c} \frac{b_2}{x'b_2+t'a_2+y\theta b_3+z\varphi a_3} + \frac{a_2+b_3}{x'(a_2+b_3)+t'(a_3+b_2)} \\ -\frac{b_3}{x'b_3+t'a_3+y\theta b_2+z\varphi a_2} \end{array}\right] \\ & +y\,(\theta-1)\left[\frac{b_3}{x'b_2+t'a_2+y\theta b_3+z\varphi a_3} - \frac{b_2}{x'b_3+t'a_3+y\theta b_2+z\varphi a_2}\right] \\ & +z\,(\varphi-1)\left[\frac{a_3}{x'b_2+t'a_2+y\theta b_3+z\varphi a_3} - \frac{a_2}{x'b_3+t'a_3+y\theta b_2+z\varphi a_2}\right] \end{aligned}$$

$$+\,(t'-t)\left[\begin{array}{c} \frac{a_2}{x'b_2+t'a_2+y\theta b_3+z\varphi a_3} + \frac{a_3+b_2}{x'(a_2+b_3)+t'(a_3+b_2)} \\ -\frac{a_3}{x'b_3+t'a_3+y\theta b_2+z\varphi a_2} \end{array}\right] \geq 0. \tag{14}$$

Since $x' > x, \theta > 1, t' > t$ and $\varphi > 1$ so the factors $x'-x, t'-t, \theta-1$ and $\varphi - 1$ can be ignored and (14) can be simplified as

$$\left[\begin{array}{c} \frac{b_2}{x'b_2+t'a_2+y\theta b_3+z\varphi a_3} + \frac{a_2+b_3}{x'(a_2+b_3)+t'(a_3+b_2)} \\ -\frac{b_3}{x'b_3+t'a_3+y\theta b_2+z\varphi a_2} \end{array}\right] \geq 0 \tag{15}$$

and

$$\left[\begin{array}{c} \frac{a_2}{x'b_2+t'a_2+y\theta b_3+z\varphi a_3} + \frac{a_3+b_2}{x'(a_2+b_3)+t'(a_3+b_2)} \\ -\frac{a_3}{x'b_3+t'a_3+y\theta b_2+z\varphi a_2} \end{array}\right] \geq 0 \tag{16}$$

In particular, setting $a_2 = a_3 = 0$ in inequality (15) implies

$$\begin{aligned} \frac{b_2}{x'b_2+y\theta b_3} + \frac{b_3}{x'b_3+t'b_2} - \frac{b_3}{x'b_3+y\theta b_2} &\geq 0 \\ \left[t'b_2 + 2x'b_3\right] y\theta b_2 + \left[y^2\theta^2 - t'y\theta + (x')^2\right] b_3^2 &\geq 0 \\ (x')^2 &\geq y\theta\,(t'-y\theta) \end{aligned} \tag{17}$$

and setting $b_2 = b_3 = 0$ in inequality (16) implies

$$\begin{aligned} \frac{a_2}{t'a_2 + z\varphi a_3} + \frac{a_3}{x'a_2 + t'a_3} - \frac{a_3}{t'a_3 + z\varphi a_2} &\geq 0 \\ \left[2t'a_3 + x'a_2\right] z\varphi a_2 + \left[z^2\varphi^2 + (t')^2 - x'z\varphi\right] a_3^2 &\geq 0 \\ (t')^2 &\geq z\varphi\left(x' - z\varphi\right) \qquad (18) \end{aligned}$$

Other combinations of $a_{i's}$ and $b_{i's}$ do not provide any other condition on x, x', t, t', y, z. Setting $a_2 = b_2 = c = d = 0$ (13) implies

$$\left[\frac{(x'-x)\,a_1 + (t'-t)\,b_1 + y\,(\theta-1)\,b_3 + z\,(\varphi-1)\,a_3}{x'a_1 + t'b_1 + y\theta b_3 + z\varphi a_3}\right] + \left[\frac{(x'-x)\,b_3 + (t'-t)\,a_3 + y\,(\theta-1)\,b_1 + z\,(\varphi-1)\,a_1}{x'b_3 + t'a_3 + y\theta b_1 + z\varphi a_1}\right] - \left[\frac{(x'-x)\,(a_1+b_3) + (t'-t)\,(a_3+b_1)}{x'\,(a_1+b_3) + t'\,(a_3+b_1)}\right] \geq 0$$

$$\begin{aligned} \Rightarrow\ (x'-x)&\left[\frac{a_1}{x'a_1+t'b_1+y\theta b_3+z\varphi a_3} + \frac{b_3}{x'b_3+t'a_3+y\theta b_1+z\varphi a_1} - \frac{a_1+b_3}{x'(a_1+b_3)+t'(a_3+b_1)}\right] \\ &+y\,(\theta-1)\left[\frac{b_3}{x'a_1 + t'b_1 + y\theta b_3 + z\varphi a_3} + \frac{b_1}{x'b_3 + t'a_3 + y\theta b_1 + z\varphi a_1}\right] \\ &+z\,(\varphi-1)\left[\frac{a_3}{x'a_1 + t'b_1 + y\theta b_3 + z\varphi a_3} + \frac{a_1}{x'b_3 + t'a_3 + y\theta b_1 + z\varphi a_1}\right] \\ &+(t'-t)\left[\frac{b_1}{x'a_1+t'b_1+y\theta b_3+z\varphi a_3} + \frac{a_3}{x'b_3+t'a_3+y\theta b_1+z\varphi a_1} - \frac{a_3+b_1}{x'(a_1+b_3)+t'(a_3+b_1)}\right] \geq 0. \end{aligned}$$

Since $x' > x, \theta > 1, t' > t$ and $\varphi > 1$ so the factors $x' - x, t' - t, \theta - 1$ and $\varphi - 1$ can be omitted and this leads to the inequality (19) and the inequality (20),

$$\left[\frac{a_1}{x'a_1+t'b_1+y\theta b_3+z\varphi a_3} + \frac{b_3}{x'b_3+t'a_3+y\theta b_1+z\varphi a_1} - \frac{a_1+b_3}{x'(a_1+b_3)+t'(a_3+b_1)}\right] \geq 0 \qquad (19)$$

and

$$\left[\frac{b_1}{x'a_1+t'b_1+y\theta b_3+z\varphi a_3} + \frac{a_3}{x'b_3+t'a_3+y\theta b_1+z\varphi a_1} - \frac{a_3+b_1}{x'(a_1+b_3)+t'(a_3+b_1)}\right] \geq 0. \qquad (20)$$

In particular, setting $a_1 = a_3 = 0$ in inequality (19) implies

$$\frac{b_3}{x'b_3 + y\theta b_1} - \frac{b_3}{x'b_3 + t'b_1} \geq 0$$

$$\begin{aligned} &\Rightarrow \quad b_3\left(x'b_3 + t'b_1\right) - b_3\left(x'b_3 + y\theta b_1\right) \geq 0 \\ &\Rightarrow \quad t' \geq y\theta \end{aligned}$$

and by setting $b_1 = b_3 = 0$ in (19) leads to

$$t' \geq z\varphi$$

That is,

$$t' \geq \max\left(y\theta, z\varphi\right) \tag{21}$$

Similarly $a_3 = b_1 = 0$ in (19) leads to the conditions $(x')^2 \geq yz\theta\varphi$ and by setting $a_1 = a_3 = 0$ and $b_1 = b_3 = 0$ in (20) implies

$$x' \geq \max\left(y\theta, z\varphi\right) \tag{22}$$

and $a_1 = b_3 = 0$ in inequality (20) gives

$$(t')^2 \geq yz\theta\varphi$$

Combining these two conditions (21) and (22) lead to

$$\min\left(x', t'\right) \geq \max\left(y\theta, z\varphi\right) \tag{23}$$

Setting $a_3 = b_3 = c = d = 0$ (13) implies

$$\begin{aligned} &\left[\frac{(x'-x)\left(a_1+b_2\right)+(t'-t)\left(a_2+b_1\right)}{x'\left(a_1+b_2\right)+t'\left(a_2+b_1\right)}\right] \\ &+\left[\frac{(x'-x)\,a_2+(t'-t)\,b_2+y\left(\theta-1\right)b_1+z\left(\varphi-1\right)a_1}{x'a_2+t'b_2+y\theta b_1+z\varphi a_1}\right] \end{aligned}$$

$$-\left[\frac{(x'-x)\,a_1+(t'-t)\,b_1+y\left(\theta-1\right)b_2+z\left(\varphi-1\right)a_2}{x'a_1+t'b_1+y\theta b_2+z\varphi a_2}\right] \geq 0$$

$$
\begin{aligned}
&(x'-x)\left[\begin{array}{c}\frac{a_1+b_2}{x'(a_1+b_2)+t'(a_2+b_1)}+\frac{a_2}{x'a_2+t'b_2+y\theta b_1+z\varphi a_1}\\ -\frac{a_1}{x'a_1+t'b_1+y\theta b_2+z\varphi a_2}\end{array}\right]\\
&+y\left(\theta-1\right)\left[\frac{b_1}{x'a_2+t'b_2+y\theta b_1+z\varphi a_1}-\frac{b_2}{x'a_1+t'b_1+y\theta b_2+z\varphi a_2}\right]\\
&+z\left(\varphi-1\right)\left[\frac{a_1}{x'a_2+t'b_2+y\theta b_1+z\varphi a_1}-\frac{a_2}{x'a_1+t'b_1+y\theta b_2+z\varphi a_2}\right]
\end{aligned}
$$

$$
+\left(t'-t\right)\left[\begin{array}{c}\frac{a_2+b_1}{x'(a_1+b_2)+t'(a_2+b_1)}+\frac{b_2}{x'a_2+t'b_2+y\theta b_1+z\varphi a_1}\\ -\frac{b_1}{x'a_1+t'b_1+y\theta b_2+z\varphi a_2}\end{array}\right]\geq 0.
$$

Since $x' > x, \theta > 1, t' > t$ and $\varphi > 1$ so the factors $x'-x, t'-t, \theta-1$ and $\varphi-1$ can be omitted and this leads to the inequality (24) and the inequality (25),

$$
\left[\begin{array}{c}\frac{a_1+b_2}{x'(a_1+b_2)+t'(a_2+b_1)}+\frac{a_2}{x'a_2+t'b_2+y\theta b_1+z\varphi a_1}\\ -\frac{a_1}{x'a_1+t'b_1+y\theta b_2+z\varphi a_2}\end{array}\right]\geq 0 \qquad (24)
$$

and

$$
\left[\begin{array}{c}\frac{a_2+b_1}{x'(a_1+b_2)+t'(a_2+b_1)}+\frac{b_2}{x'a_2+t'b_2+y\theta b_1+z\varphi a_1}\\ -\frac{b_1}{x'a_1+t'b_1+y\theta b_2+z\varphi a_2}\end{array}\right]\geq 0. \qquad (25)
$$

In particular, setting $a_1 = a_2 = 0$ in the inequality (25) implies

$$
\begin{aligned}
\frac{b_1}{x'b_2+t'b_1}+\frac{b_2}{t'b_2+y\theta b_1}-\frac{b_1}{t'b_1+y\theta b_2} &\geq 0\\
\left[x'b_2+2t'b_1\right]y\theta b_2+\left[(t')^2-x'y\theta+y^2\theta^2\right]b_1^2 &\geq 0\\
(t')^2 &\geq y\theta\left(x'-y\theta\right). \qquad (26)
\end{aligned}
$$

In particular, setting $b_1 = b_2 = 0$ in the inequality (24) implies

$$
\begin{aligned}
\frac{a_1}{x'a_1+t'a_2}+\frac{a_2}{x'a_2+z\varphi a_1}-\frac{a_1}{x'a_1+z\varphi a_2} &\geq 0\\
\left[2x'a_1+t'a_2\right]z\varphi a_2+\left[z^2\varphi^2-t'z\varphi+(x')^2\right]a_1^2 &\geq 0\\
(x')^2 &\geq z\varphi\left(t'-z\varphi\right). \qquad (27)
\end{aligned}
$$

Other combinations of $a_{i's}$ and $b_{i's}$ do not provide any other condition on x, x', t, t', y, z. Above conditions (17) and (27) can be combine to

$$(x')^2 \geq \max(y\theta\,(t' - y\theta)\,, z\varphi\,(t' - z\varphi))$$

and conditions (18) and (26) can be combine to

$$(t')^2 \geq \max(y\theta\,(x' - y\theta)\,, z\varphi\,(x' - z\varphi)).$$

Thus

$$\begin{aligned} \min(\,x', t') &\geq \max(y\theta, z\varphi) \\ \wedge(x')^2 &\geq \max(y\theta\,(t' - y\theta)\,, z\varphi\,(t' - z\varphi)) \\ \wedge(t')^2 &\geq \max(y\theta\,(x' - y\theta)\,, z\varphi\,(x' - z\varphi)) \end{aligned}$$

are the required necessary conditions for family to be W-transitive.

3 Transitivity of Similarity Measures

A common approach for comparing two objects is to select an appropriate list of features then prepare a binary vector of $\{0, 1\}$ based on the fact that if the object has certain feature will be recorded as 1 and 0 otherwise. The degree of similarity of the two objects is then expressed in terms of cardinalities of these binary vector sets A and B in a finite universe X as

$$S(A, B) = \frac{x\alpha_{A,B} + t\omega_{A,B} + y\delta_{A,B} + z\nu_{A,B}}{x'\alpha_{A,B} + t'\omega_{A,B} + y'\delta_{A,B} + z'\nu_{A,B}} \tag{28}$$

with $A, B \in P(X), \omega_{A,B} = \max(|A\backslash B|\,, |B\backslash A|)\,, \alpha_{A,B} = \min(|A\backslash B|\,, |B\backslash A|)\,,$ $\delta_{A,B} = |A \cap B|$, $\nu_{A,B} = |(A \cup B)^c|$ and parameters $x, x', t, t', y, y', z, z' \in \{0, 1\}$. The similarity measure is symmetric, that is, for all $A, B \in F(X)$, $S(A, B) = S(B, A)$ and fuzzy reflexive for any $A \in F(X), S(A, A) \in [0, 1]$.

Janssens et al [34] used four parameters to test the various properties of monotonicity and transitivity for Łukasiewicz, Product and Min t-norms. Some members of the family (28) with parameters discussed by Janssens [34] in the light of eight parameters are given in Table 2. Many other measures cannot be discussed under parametric family of similarity measures using four parameters, some of them are given in Table 3. Some of these measures are explored in this section by using all eight parameters. Fuzzy similarity

Tab. 2: Some members of Family (28) with 4 parameters.

Measure	Expression	x	x'	t	t'	y	y'	z	z'
Jaccard [28]	$\frac{\lvert A\cap B\rvert}{\lvert A\cup B\rvert}$	0	1	0	1	1	1	0	0
Simple Matching [29]	$1-\frac{\lvert A\triangle B\rvert}{n}$	0	1	0	1	1	1	1	1
Dice [30]	$\frac{2\lvert A\cap B\rvert}{\lvert A\triangle B\rvert+2\lvert A\cap B\rvert}$	0	1	0	1	2	2	0	0
Rogers & Tanimoto [31]	$\frac{n-\lvert A\triangle B\rvert}{n+\lvert A\triangle B\rvert}$	0	2	0	2	1	1	1	1
Sokal & Sneath 1 [32]	$\frac{\lvert A\cap B\rvert}{2\lvert A\triangle B\rvert+\lvert A\cap B\rvert}$	0	2	0	2	1	1	0	0
Sokal & Sneath 2 [33]	$1-\frac{\lvert A\triangle B\rvert}{2n-\lvert A\triangle B\rvert}$	0	1	0	1	2	2	2	2

measure will hold fuzzy reflexivity for $y' = y\theta, z' = z\varphi$ with $\theta, \varphi \geq 1$, and the above relation of similarity measure can be written as

$$S(A,B) = \frac{x\alpha_{A,B} + t\omega_{A,B} + y\delta_{A,B} + z\nu_{A,B}}{x'\alpha_{A,B} + t'\omega_{A,B} + y\theta\delta_{A,B} + z\varphi\nu_{A,B}} \tag{29}$$

For simplicity, if we use only four parameters, that is, $x = t, x' = t'$, the above relation becomes

$$S(A,B) = \frac{x \triangle_{A,B} + y\delta_{A,B} + z\nu_{A,B}}{x' \triangle_{A,B} + y\theta\delta_{A,B} + z\varphi\nu_{A,B}} \tag{30}$$

which is an extension of four parameters of Janssens' to six parameters

$$S(A,B) = \frac{x \triangle_{A,B} + y\delta_{A,B} + z\nu_{A,B}}{x' \triangle_{A,B} + y\delta_{A,B} + z\nu_{A,B}} \tag{31}$$

Tab. 3: Some members of Family (28) with 8 parameters.

Measure	Expression	x	x'	t	t'	y	y'	z	z'
Sokal & Sneath 3	$\frac{n}{\lvert A\triangle B\rvert}-1$	0	1	0	1	1	0	1	0
Faith [35]	$1-\frac{2\lvert A\triangle B\rvert+\lvert (A\cup B)^c\rvert}{2n}$	0	2	0	2	2	2	1	2
Kulczynski 1	$\frac{\lvert A\cap B\rvert}{\lvert A\triangle B\rvert}$	0	1	0	1	1	0	0	0
Russell & Rao [29]	$\frac{\lvert A\cap B\rvert}{n}$	0	1	0	1	1	1	0	1

These similarity measures are symmetric for any fuzzy subsets A, B and will be fuzzy reflexive if $y' = y\theta$ and $z' = z\varphi$ for $\theta, \varphi \geq 1$. That is with positive real parameters x, x', t, t', y, z. $S(A,B) \in [0,1]$ if $0 \leq x < x'$ and $0 \leq t < t'$. This section is focused to characterize the Łukasiewicz-transitive members of family (28).

Theorem 3 The W-transitive members of family (30) are characterized by the necessary condition

$$x' \geq \max(y\theta, z\varphi).$$

Proof: To determine the conditions on parameters x, x', y, and z for transitivity, the similarity measure S is W-transitive if it holds that

$$W\left(S(A,B), S(B,C)\right) \leq S(A,C)$$

$$S(A,B) + S(B,C) - 1 \leq S(A,C)$$

Applying $N(x) = 1 - x$ to both side of above inequality, we have

$$[1 - S(A,B)] + [1 - S(B,C)] - [1 - S(A,C)] \geq 0$$

In terms of family (30), the above inequality can be written as

$$\left[1 - \frac{x \triangle_{A,B} + y\delta_{A,B} + z\nu_{A,B}}{x' \triangle_{A,B} + y\theta\delta_{A,B} + z\varphi\nu_{A,B}}\right] + \left[1 - \frac{x \triangle_{B,C} + y\delta_{B,C} + z\nu_{B,C}}{x' \triangle_{B,C} + y\theta\delta_{B,C} + z\varphi\nu_{B,C}}\right]$$

$$- \left[1 - \frac{x \triangle_{A,C} + y\delta_{A,C} + z\nu_{A,C}}{x' \triangle_{A,C} + y\theta\delta_{A,C} + z\varphi\nu_{A,C}}\right] \geq 0$$

$$\left[\frac{(x'-x) \triangle_{A,B} + y(\theta-1)\delta_{A,B} + z(\varphi-1)\nu_{A,B}}{x' \triangle_{A,B} + y\theta\delta_{A,B} + z\varphi\nu_{A,B}}\right] + \left[\frac{(x'-x) \triangle_{B,C} + y(\theta-1)\delta_{B,C} + z(\varphi-1)\nu_{B,C}}{x' \triangle_{B,C} + y\theta\delta_{B,C} + z\varphi\nu_{B,C}}\right]$$

$$- \left[\frac{(x'-x) \triangle_{A,C} + y(\theta-1)\delta_{A,C} + z(\varphi-1)\nu_{A,C}}{x' \triangle_{A,C} + y\theta\delta_{A,C} + z\varphi\nu_{A,C}}\right] \geq 0$$

Let us assume that $\triangle_{A,B} = |A \triangle B|$, $\delta_{A,B} = |A \cap B|$, $\nu_{A,B} = |(A \cup B)^c|$. Similarly, $\triangle_{B,C} = |B \triangle C|$, $\delta_{B,C} = |B \cap C|$, $\nu_{B,C} = |(B \cup C)^c|$, and $\triangle_{A,C} = |A \triangle C|$, $\delta_{A,C} = |A \cap C|$, $\nu_{A,C} = |(A \cup C)^c|$, then above inequality becomes

$$\left[\frac{(x'-x)\,|A\vartriangle B|+y\,(\theta-1)\,|A\cap B|+z\,(\varphi-1)\,|(A\cup B)^c|}{x'\,|A\vartriangle B|+y\theta\,|A\cap B|+z\varphi\,|(A\cup B)^c|}\right]$$
$$+\left[\frac{(x'-x)\,|B\vartriangle C|+y\,(\theta-1)\,|B\cap C|+z\,(\varphi-1)\,|(B\cup C)^c|}{x'\,|B\vartriangle C|+y\theta\,|B\cap C|+z\varphi\,|(B\cup C)^c|}\right]$$

$$-\left[\frac{(x'-x)\,|A\vartriangle C|+y\,(\theta-1)\,|A\cap C|+z\,(\varphi-1)\,|(A\cup C)^c|}{x'\,|A\vartriangle C|+y\theta\,|A\cap C|+z\varphi\,|(A\cup C)^c|}\right]\geq 0$$

Consider the setting in figure, and substituting the cardinalities (5), the above inequality implies

$$\left[\frac{(x'-x)\,(a_1+b_2+a_2+b_1)+y\,(\theta-1)\,(b_3+c)+z\,(\varphi-1)\,(a_3+d)}{x'\,(a_1+b_2+a_2+b_1)+y\theta\,(b_3+c)+z\varphi\,(a_3+d)}\right]$$
$$+\left[\frac{(x'-x)\,(a_2+b_3+a_3+b_2)+y\,(\theta-1)\,(b_1+c)+z\,(\varphi-1)\,(a_1+d)}{x'\,(a_2+b_3+a_3+b_2)+y\theta\,(b_1+c)+z\varphi\,(a_1+d)}\right]$$

$$-\left[\frac{(x'-x)\,(a_1+b_3+a_3+b_1)+y\,(\theta-1)\,(b_2+c)+z\,(\varphi-1)\,(a_2+d)}{x'\,(a_1+b_3+a_3+b_1)+y\theta\,(b_2+c)+z\varphi\,(a_2+d)}\right]\geq 0 \tag{32}$$

Similarly setting $a_2=b_2=c=d=0$, the inequality (32) implies that

$$\left[\frac{(x'-x)\,(a_1+b_1)+y\,(\theta-1)\,b_3+z\,(\varphi-1)\,a_3}{x'\,(a_1+b_1)+y\theta b_3+z\varphi a_3}\right]$$

$$+\left[\frac{(x'-x)\,(b_3+a_3)+y\,(\theta-1)\,b_1+z\,(\varphi-1)\,a_1}{x'\,(b_3+a_3)+y\theta b_1+z\varphi a_1}\right]-\left[\frac{(x'-x)}{x'}\right]\geq 0$$

$$\Rightarrow (x'-x)\left[\frac{b_1+a_1}{x'\,(a_1+b_1)+y\theta b_3+z\varphi a_3}+\frac{b_3+a_3}{x'\,(b_3+a_3)+y\theta b_1+z\varphi a_1}-\frac{1}{x'}\right]$$

$$+y\,(\theta-1)\left[\frac{b_3}{x'\,(a_1+b_1)+y\theta b_3+z\varphi a_3}+\frac{b_1}{x'\,(b_3+a_3)+y\theta b_1+z\varphi a_1}\right]$$

$$+z(\varphi-1)\left[\frac{a_3}{x'(a_1+b_1)+y\theta b_3+z\varphi a_3}+\frac{a_1}{x'(b_3+a_3)+y\theta b_1+z\varphi a_1}\right]\geq 0$$

Since $x'>x, \theta>1$ and $\varphi>1$ so the factors $x'-x, \theta-1$ and $\varphi-1$ can be omitted and we obtain

$$\frac{b_1+a_1}{x'(a_1+b_1)+y\theta b_3+z\varphi a_3}+\frac{b_3+a_3}{x'(b_3+a_3)+y\theta b_1+z\varphi a_1}-\frac{1}{x'}\geq 0 \qquad (33)$$

In particular, setting $a_1=a_3=0$, the inequality (33) leads to the results

$$\begin{aligned}\frac{b_1}{x'b_1+y\theta b_3}+\frac{b_3}{x'b_3+y\theta b_1}-\frac{1}{x'} &\geq 0\\ b_1b_3\left((x')^2-y^2\theta^2\right) &\geq 0\end{aligned}$$

which is fulfilled only if $x'\geq y\theta$ and setting $b_1=b_3=0$, the inequality (33) leads to the results

$$\begin{aligned}\frac{a_1}{x'a_1+z\varphi a_3}+\frac{a_3}{x'a_3+z\varphi a_1}-\frac{1}{x'} &\geq 0\\ a_1a_3\left((x')^2-z^2\varphi^2\right) &\geq 0\end{aligned}$$

which is satisfied if $x'\geq z\varphi$. Similarly, setting $a_1=b_3=0$, in the inequality (33) as

$$\frac{b_1}{x'b_1+z\varphi a_3}-\frac{1}{x'}+\frac{a_3}{x'a_3+y\theta b_1}\geq 0$$

and $b_1=a_3=0$, in the inequality (33) as

$$\frac{a_1}{x'a_1+y\theta b_3}-\frac{1}{x'}+\frac{b_3}{x'b_3+z\varphi a_1}\geq 0$$

lead to the result

$$(x')^2\geq yz\theta\varphi$$

These conditions can be combine into a single condition as

$$x'\geq\max(y\theta,z\varphi). \qquad (34)$$

Setting $a_1 = b_1 = c = d = 0$, and $a_3 = b_3 = c = d = 0$ the inequality (32) does not imply any further condition on x, x', y and z. Thus (34) is the necessary condition for transitivity.

Theorem 4 The W-transitive members of family (29) are characterized by the necessary condition

$$\begin{aligned} \min(x', t') &\geq \max(y\theta, z\varphi) \\ \wedge(t')^2 &\geq \max((x' - y\theta)\, y\theta, (x' - z\varphi)\, z\varphi) \\ \wedge(x')^2 &\geq \max((t' - y\theta)\, y\theta, (t' - z\varphi)\, z\varphi) \end{aligned}$$

Proof: To determine the conditions on parameters x, x', t, t', y, and z for transitivity, the similarity measure S is W-transitive if it holds that

$$W(S(A, B), S(B, C)) \leq S(A, C)$$

$$S(A, B) + S(B, C) - 1 \leq S(A, C)$$

Applying $N(x) = 1 - x$ to both side of above inequality, we have

$$[1 - S(A, B)] + [1 - S(B, C)] - [1 - S(A, C)] \geq 0$$

In terms of family (29), the above inequality can be written as

$$\begin{aligned} &\left[1 - \frac{x\alpha_{A,B} + t\omega_{A,B} + y\delta_{A,B} + z\nu_{A,B}}{x'\alpha_{A,B} + t'\omega_{A,B} + y\theta\delta_{A,B} + z\varphi\nu_{A,B}}\right] \\ &+ \left[1 - \frac{x\alpha_{B,C} + t\omega_{B,C} + y\delta_{B,C} + z\nu_{B,C}}{x'\alpha_{B,C} + t'\omega_{B,C} + y\theta\delta_{B,C} + z\varphi\nu_{B,C}}\right] \end{aligned}$$

$$-\left[1 - \frac{x\alpha_{A,C} + t\omega_{A,C} + y\delta_{A,C} + z\nu_{A,C}}{x'\alpha_{A,C} + t'\omega_{A,C} + y\theta\delta_{A,C} + z\varphi\nu_{A,C}}\right] \geq 0$$

Let us assume that

$$\begin{aligned} \alpha_{A,B} &= \min(|A \backslash B|, |B \backslash A|) = |A \backslash B|, \\ \omega_{A,B} &= \max(|A \backslash B|, |B \backslash A|) = |B \backslash A|, \\ \delta_{A,B} &= |A \cap B|, \\ \nu_{A,B} &= |(A \cup B)^c|. \end{aligned}$$

Similarly,

$$\begin{aligned}\alpha_{B,C} &= \min\left(\left|B\backslash C\right|,\left|C\backslash B\right|\right)=\left|B\backslash C\right|,\\ \omega_{B,C} &= \max\left(\left|B\backslash C\right|,\left|C\backslash B\right|\right)=\left|C\backslash B\right|,\\ \delta_{B,C} &= \left|B\cap C\right|,\\ \nu_{B,C} &= \left|(B\cup C)^{c}\right|,\end{aligned}$$

and

$$\begin{aligned}\alpha_{A,C} &= \min\left(\left|A\backslash C\right|,\left|C\backslash A\right|\right)=\left|A\backslash C\right|,\\ \omega_{A,C} &= \max\left(\left|A\backslash C\right|,\left|C\backslash A\right|\right)=\left|C\backslash A\right|,\\ \delta_{A,C} &= \left|A\cap C\right|,\\ \nu_{A,C} &= \left|(A\cup C)^{c}\right|,\end{aligned}$$

then above inequality becomes

$$\begin{aligned}&\left[1-\frac{x\left|A\backslash B\right|+t\left|B\backslash A\right|+y\left|A\cap B\right|+z\left|(A\cup B)^{c}\right|}{x'\left|A\backslash B\right|+t'\left|B\backslash A\right|+y\theta\left|A\cap B\right|+z\varphi\left|(A\cup B)^{c}\right|}\right]\\ &+\left[1-\frac{x\left|B\backslash C\right|+t\left|C\backslash B\right|+y\left|B\cap C\right|+z\left|(B\cup C)^{c}\right|}{x'\left|B\backslash C\right|+t'\left|C\backslash B\right|+y\theta\left|B\cap C\right|+z\varphi\left|(B\cup C)^{c}\right|}\right]\\ &-\left[1-\frac{x\left|A\backslash C\right|+t\left|C\backslash A\right|+y\left|A\cap C\right|+z\left|(A\cup C)^{c}\right|}{x'\left|A\backslash C\right|+t'\left|C\backslash A\right|+y\theta\left|A\cap C\right|+z\varphi\left|(A\cup C)^{c}\right|}\right]\geq 0.\end{aligned}$$

Consider the setting in figure, and substituting the cardinalities (5), the above inequality implies

$$\begin{aligned}&\left[\frac{(x'-x)(a_1+b_2)+(t'-t)(a_2+b_1)+y(\theta-1)(b_3+c)+z(\varphi-1)(a_3+d)}{x'(a_1+b_2)+t'(a_2+b_1)+y\theta(b_3+c)+z\varphi(a_3+d)}\right]\\ &+\left[\frac{(x'-x)(a_2+b_3)+(t'-t)(a_3+b_2)+y(\theta-1)(b_1+c)+z(\varphi-1)(a_1+d)}{x'(a_2+b_3)+t'(a_3+b_2)+y\theta(b_1+c)+z\varphi(a_1+d)}\right]\\ &-\left[\frac{(x'-x)(a_1+b_3)+(t'-t)(a_3+b_1)+y(\theta-1)(b_2+c)+z(\varphi-1)(a_2+d)}{x'(a_1+b_3)+t'(a_3+b_1)+y\theta(b_2+c)+z\varphi(a_2+d)}\right]\geq 0.\end{aligned}\tag{35}$$

Similarly setting $a_1=b_1=c=d=0$, the inequality (35) implies that

$$\left[\frac{(x'-x)\,b_2+(t'-t)\,a_2+y(\theta-1)\,b_3+z(\varphi-1)\,a_3}{x'b_2+t'a_2+y\theta b_3+z\varphi a_3}\right]$$

$$+\left[\frac{(x'-x)(a_2+b_3)+(t'-t)(a_3+b_2)}{x'(a_2+b_3)+t'(a_3+b_2)}\right]$$
$$-\left[\frac{(x'-x)b_3+(t'-t)a_3+y(\theta-1)b_2+z(\varphi-1)a_2}{x'b_3+t'a_3+y\theta b_2+z\varphi a_2}\right]\geq 0$$

$$\begin{aligned}
&(x'-x)\left[\begin{array}{c}\frac{b_2}{x'b_2+t'a_2+y\theta b_3+z\varphi a_3}+\frac{a_2+b_3}{x'(a_2+b_3)+t'(a_3+b_2)}\\ -\frac{b_3}{x'b_3+t'a_3+y\theta b_2+z\varphi a_2}\end{array}\right]\\
&+y(\theta-1)\left[\frac{b_3}{x'b_2+t'a_2+y\theta b_3+z\varphi a_3}-\frac{b_2}{x'b_3+t'a_3+y\theta b_2+z\varphi a_2}\right]\\
&+z(\varphi-1)\left[\frac{a_3}{x'b_2+t'a_2+y\theta b_3+z\varphi a_3}-\frac{a_2}{x'b_3+t'a_3+y\theta b_2+z\varphi a_2}\right]\\
&+(t'-t)\left[\begin{array}{c}\frac{a_2}{x'b_2+t'a_2+y\theta b_3+z\varphi a_3}+\frac{a_3+b_2}{x'(a_2+b_3)+t'(a_3+b_2)}\\ -\frac{a_3}{x'b_3+t'a_3+y\theta b_2+z\varphi a_2}\end{array}\right]\geq 0.
\end{aligned}$$

Since $x'>x,\theta>1$ and $\varphi>1$ so the factors $x'-x,\theta-1$ and $\varphi-1$ can be omitted and we obtain

$$\left[\begin{array}{c}\frac{b_2}{x'b_2+t'a_2+y\theta b_3+z\varphi a_3}+\frac{a_2+b_3}{x'(a_2+b_3)+t'(a_3+b_2)}\\ -\frac{b_3}{x'b_3+t'a_3+y\theta b_2+z\varphi a_2}\end{array}\right]\geq 0 \tag{36}$$

and

$$\left[\begin{array}{c}\frac{a_2}{x'b_2+t'a_2+y\theta b_3+z\varphi a_3}+\frac{a_3+b_2}{x'(a_2+b_3)+t'(a_3+b_2)}\\ -\frac{a_3}{x'b_3+t'a_3+y\theta b_2+z\varphi a_2}\end{array}\right]\geq 0 \tag{37}$$

In particular, setting $a_2=a_3=0$, the inequality (36) leads to the results

$$\begin{aligned}
\frac{b_2}{x'b_2+y\theta b_3}+\frac{b_3}{x'b_3+t'b_2}-\frac{b_3}{x'b_3+y\theta b_2} &\geq 0\\
[2x'b_3+t'b_2]\,y\theta b_2+\left[(x')^2-(t'-y\theta)\,y\theta\right]b_3^2 &\geq 0\\
(x')^2 &\geq (t'-y\theta)\,y\theta \qquad (38)
\end{aligned}$$

and setting $b_2=b_3=0$, the inequality (37) leads to the results

$$\begin{aligned}
\frac{a_2}{t'a_2+z\varphi a_3}+\frac{a_3}{x'a_2+t'a_3}-\frac{a_3}{t'a_3+z\varphi a_2} &\geq 0\\
[x'a_2+2t'a_3]\,z\varphi a_2+\left[z^2\varphi^2-x'z\varphi+(t')^2\right]a_3^2 &\geq 0\\
(t')^2 &\geq (x'-z\varphi)\,z\varphi. \qquad (39)
\end{aligned}$$

Other combinations of $a_{i's}$ and $b_{i's}$ do not provide any other condition on x, x', t, t', y, z. Similarly setting $a_2 = b_2 = c = d = 0$, the inequality (35) implies that

$$\left[\frac{(x'-x)\,a_1 + (t'-t)\,b_1 + y\,(\theta-1)\,b_3 + z\,(\varphi-1)\,a_3}{x'a_1 + t'b_1 + y\theta b_3 + z\varphi a_3}\right] + \left[\frac{(x'-x)\,b_3 + (t'-t)\,a_3 + y\,(\theta-1)\,b_1 + z\,(\varphi-1)\,a_1}{x'b_3 + t'a_3 + y\theta b_1 + z\varphi a_1}\right]$$

$$-\left[\frac{(x'-x)\,(a_1+b_3) + (t'-t)\,(a_3+b_1)}{x'\,(a_1+b_3) + t'\,(a_3+b_1)}\right] \geq 0$$

$$\begin{aligned}&(x'-x)\left[\begin{array}{c}\frac{a_1}{x'a_1+t'b_1+y\theta b_3+z\varphi a_3} + \frac{b_3}{x'b_3+t'a_3+y\theta b_1+z\varphi a_1}\\ -\frac{a_1+b_3}{x'(a_1+b_3)+t'(a_3+b_1)}\end{array}\right]\\ &+y\,(\theta-1)\left[\frac{b_3}{x'a_1+t'b_1+y\theta b_3+z\varphi a_3} + \frac{b_1}{x'b_3+t'a_3+y\theta b_1+z\varphi a_1}\right]\\ &+z\,(\varphi-1)\left[\frac{a_3}{x'a_1+t'b_1+y\theta b_3+z\varphi a_3} + \frac{a_1}{x'b_3+t'a_3+y\theta b_1+z\varphi a_1}\right]\end{aligned}$$

$$+(t'-t)\left[\begin{array}{c}\frac{b_1}{x'a_1+t'b_1+y\theta b_3+z\varphi a_3} + \frac{a_3}{x'b_3+t'a_3+y\theta b_1+z\varphi a_1}\\ -\frac{a_3+b_1}{x'(a_1+b_3)+t'(a_3+b_1)}\end{array}\right] \geq 0.$$

Since $x' > x, \theta > 1, t' > t$ and $\varphi > 1$ so the factors $x'-x, t'-t, \theta-1$ and $\varphi-1$ can be ignored and above inequality leads to

$$\left[\begin{array}{c}\frac{a_1}{x'a_1+t'b_1+y\theta b_3+z\varphi a_3} + \frac{b_3}{x'b_3+t'a_3+y\theta b_1+z\varphi a_1}\\ -\frac{a_1+b_3}{x'(a_1+b_3)+t'(a_3+b_1)}\end{array}\right] \geq 0 \tag{40}$$

and

$$\left[\begin{array}{c}\frac{b_1}{x'a_1+t'b_1+y\theta b_3+z\varphi a_3} + \frac{a_3}{x'b_3+t'a_3+y\theta b_1+z\varphi a_1}\\ -\frac{a_3+b_1}{x'(a_1+b_3)+t'(a_3+b_1)}\end{array}\right] \geq 0. \tag{41}$$

In particular, setting $a_1 = a_3 = 0$ and $b_1 = b_3 = 0$ the inequality (40) leads to the results

$$\begin{aligned}\frac{b_3}{x'b_3+y\theta b_1} - \frac{b_3}{x'b_3+t'b_1} &\geq 0\\ t' &\geq y\theta\end{aligned}$$

and

$$\begin{aligned}\frac{a_1}{x'a_1+z\varphi a_3}-\frac{a_1}{x'a_1+t'a_3} &\geq 0\\ t' &\geq z\varphi\end{aligned}$$

respectively, which can be written in the form

$$t' \geq \max\left(y\theta, z\varphi\right). \tag{42}$$

Also setting $a_1 = a_3 = 0$ and $b_1 = b_3 = 0$, the inequality (41) leads to the results

$$\begin{aligned}\frac{b_1}{t'b_1+y\theta b_3}-\frac{b_1}{x'b_3+t'b_1} &\geq 0\\ x' &\geq y\theta\end{aligned}$$

and

$$\begin{aligned}\frac{a_3}{t'a_3+z\varphi a_1}-\frac{a_3}{x'a_1+t'a_3} &\geq 0\\ x' &\geq z\varphi\end{aligned}$$

respectively, which can be combine in the form

$$x' \geq \max\left(y\theta, z\varphi\right) \tag{43}$$

The results (42) and (43) leads to

$$\min\left(x', t'\right) \geq \max\left(y\theta, z\varphi\right) \tag{44}$$

Similarly setting $a_3 = b_3 = c = d = 0$, the inequality (35) implies that

$$\begin{aligned}&\left[\frac{(x'-x)(a_1+b_2)+(t'-t)(a_2+b_1)}{x'(a_1+b_2)+t'(a_2+b_1)}\right]\\ &\quad+\left[\frac{(x'-x)a_2+(t'-t)b_2+y(\theta-1)b_1+z(\varphi-1)a_1}{x'a_2+t'b_2+y\theta b_1+z\varphi a_1}\right]\\ &-\left[\frac{(x'-x)a_1+(t'-t)b_1+y(\theta-1)b_2+z(\varphi-1)a_2}{x'a_1+t'b_1+y\theta b_2+z\varphi a_2}\right]\geq 0\end{aligned}$$

$$(x'-x)\left[\begin{array}{c}\frac{a_1+b_2}{x'(a_1+b_2)+t'(a_2+b_1)}+\frac{a_2}{x'a_2+t'b_2+y\theta b_1+z\varphi a_1}\\ -\frac{a_1}{x'a_1+t'b_1+y\theta b_2+z\varphi a_2}\end{array}\right]$$
$$+y(\theta-1)\left[\frac{b_1}{x'a_2+t'b_2+y\theta b_1+z\varphi a_1}-\frac{b_2}{x'a_1+t'b_1+y\theta b_2+z\varphi a_2}\right]$$
$$+z(\varphi-1)\left[\frac{a_1}{x'a_2+t'b_2+y\theta b_1+z\varphi a_1}-\frac{a_2}{x'a_1+t'b_1+y\theta b_2+z\varphi a_2}\right]$$

$$+(t'-t)\left[\begin{array}{c}\frac{a_2+b_1}{x'(a_1+b_2)+t'(a_2+b_1)}+\frac{b_2}{x'a_2+t'b_2+y\theta b_1+z\varphi a_1}\\ -\frac{b_1}{x'a_1+t'b_1+y\theta b_2+z\varphi a_2}\end{array}\right]\geq 0.$$

Since $x'>x, t'>t, \theta>1$ and $\varphi>1$ so the factors $x'-x, t'-t, \theta-1$ and $\varphi-1$ can be ignored and above inequality leads to

$$\left[\begin{array}{c}\frac{a_1+b_2}{x'(a_1+b_2)+t'(a_2+b_1)}+\frac{a_2}{x'a_2+t'b_2+y\theta b_1+z\varphi a_1}\\ -\frac{a_1}{x'a_1+t'b_1+y\theta b_2+z\varphi a_2}\end{array}\right]\geq 0 \tag{45}$$

and

$$\left[\begin{array}{c}\frac{a_2+b_1}{x'(a_1+b_2)+t'(a_2+b_1)}+\frac{b_2}{x'a_2+t'b_2+y\theta b_1+z\varphi a_1}\\ -\frac{b_1}{x'a_1+t'b_1+y\theta b_2+z\varphi a_2}\end{array}\right]\geq 0. \tag{46}$$

In particular, setting $a_1=a_2=0$, the inequality (46) leads to the results

$$\begin{aligned}\frac{b_1}{x'b_2+t'b_1}+\frac{b_2}{t'b_2+y\theta b_1}-\frac{b_1}{t'b_1+y\theta b_2} &\geq 0\\ \left[2t'b_1+x'b_2\right]y\theta b_2+\left[(t')^2-x'y\theta+y^2\theta^2\right]b_1^2 &\geq 0\\ (t')^2 &\geq (x'-y\theta)\,y\theta \qquad (47)\end{aligned}$$

and setting $b_1=b_2=0$, the inequality (45) leads to the results

$$\begin{aligned}\frac{a_1}{x'a_1+t'a_2}+\frac{a_2}{x'a_2+z\varphi a_1}-\frac{a_1}{x'a_1+z\varphi a_2} &\geq 0\\ \left[z^2\varphi^2+(x')^2-t'z\varphi\right]a_1^2+\left[t'a_2+2x'a_1\right]z\varphi a_2 &\geq 0\\ (x')^2 &\geq (t'-z\varphi)\,z\varphi. \qquad (48)\end{aligned}$$

Now results (38) and (48) leads to

$$(x')^2\geq\max\left((t'-y\theta)\,y\theta,(t'-z\varphi)\,z\varphi\right) \tag{49}$$

and results (39) and (47) merged to

$$(t')^2 \geq \max\left(\left(x' - y\theta\right) y\theta, \left(x' - z\varphi\right) z\varphi\right). \tag{50}$$

Thus the necessary condition for transitivity is obtained by (44, 49 and 50) as

$$\begin{aligned} \min\left(x', t'\right) &\geq \max\left(y\theta, z\varphi\right) \\ \wedge (t')^2 &\geq \max\left(\left(x' - y\theta\right) y\theta, \left(x' - z\varphi\right) z\varphi\right) \\ \wedge (x')^2 &\geq \max\left(\left(t' - y\theta\right) y\theta, \left(t' - z\varphi\right) z\varphi\right). \end{aligned} \tag{51}$$

Example

Suppose we want to compare three objects A, B, and C based on certain features represented by binary vectors.

$$\begin{aligned} A &: 1\,0\,1\,0\,1\,0\,1\,0\,1\,1\,0\,1 \\ B &: 0\,1\,0\,1\,1\,0\,0\,1\,0\,1\,0\,1 \\ C &: 1\,0\,0\,1\,0\,1\,1\,0\,1\,0\,1\,1 \end{aligned}$$

It is possible to find the values of the parameters which satisfy the necessary conditions and the Łukasiewicz similarity and inclusion measures are transitive. For example, putting $x = 3, x' = 10, y = 2, y' = 6, z = 4, z' = 7, t = 2$ and $t' = 10$ leads to the following similarity measure:

$$S(A, B) =$$

$$\frac{3min(|A\backslash B|, |B\backslash A|) + 2max(|A\backslash B|, |B\backslash A|) + 2|A \cap B| + 4|(A \cup B)^c|}{10min(|A\backslash B|, |B\backslash A|) + 10max(|A\backslash B|, |B\backslash A|) + 6|A \cap B| + 7|(A \cup B)^c|}.$$

It is easy to verify that the necessary conditions (51) are fulfilled and the similarity measure is transitive also. Jaccard coefficient of similarity J(A,B) and similarity measured by parametric family of cardinality based for four parameters [18, See Eq. 2.4] is measured to 0.30. The degree of similarity measure between objects A and B using 8 and 6 parameters is also 0.303922 and 0.30 respectively for the values of the parameters given in the Table 4. By assigning different weights to these parameters leads to different degree of similarity. So if we are more concerned about the absent features, we can assign more weight to them and can find the similarity between them.

Tab. 4: Degree of similarity measures S (29), S_1 (30) and S_2 (31).

	x	x'	y	y'	z	z'	t	t'	Degree of Similarity
$S(A,B)$	0	1	1	–	0	–	–	–	0.3
$S(B,C)$	0	1	1	–	0	–	–	–	0.1818
$S(A,C)$	0	1	1	–	0	–	–	–	0.4
$S_1(A,B)$	3	10	2	6	4	7	2	10	0.303922
$S_1(B,C)$	3	10	2	6	4	7	2	10	0.275229
$S_1(A,C)$	3	10	2	6	4	7	2	10	0.316327
$S_2(A,B)$	3	11	1	3	3	7	–	–	0.3
$S_2(B,C)$	3	11	1	3	3	7	–	–	0.285714
$S_2(A,C)$	3	11	1	3	3	7	–	–	0.304348

4 Conclusion

In this paper, we try to tackle the problem arises due the limitation imposed by four parameters and try to discuss the necessary conditions for transitivity of similarity and inclusion measure. This would help to compare objects based not only on positive matching but also on the basis of negative matchings. This will play a significance role medicines, image processing, text mining, etc. where negative match has a vital role. For future directions, similar parametric family can be devised for multiplication of cardinalities.

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