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Linear programming model for solution of matrix game with payoffs trapezoidal intuitionistic fuzzy number

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Abstract

In this work, we considered two-person zero-sum games with fuzzy payoffs and matrix games with payoffs of trapezoidal intuitionistic fuzzy numbers (TrIFNs). The concepts of TrIFNs and their arithmetic operations were used. The cut-set based method for matrix games with payoffs of TrIFNs was also considered. Compute the interval-type value of any alfa-constrategies by simplex method for linear programming. The proposed method is illustrated with a numerical example.

Keywords: Intuitionistic fuzzy set, matrix game, linear programming.

1 Introduction

The concept of an intuitionistic fuzzy set was proposed by Atanassov in 1986 [1]. This concept refered to the reflect of the relation among " 1 minus the degree of membership","the degree of non-membership" and " the degree of hesitation". The intuitionistic fuzzy set was rasterized by the degree of

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membership and the degree of non-membership. The intuitionistic fuzzy set had more abundant and flexible than the fuzzy set with uncertain information. Ishibuchi and Tanaka [2], Chanas and Kuchta [3], studies multiobjective programming in optimization of interval objective functions, solving interval-valued objective optimization problems. More specifically, interval-valued objective optimization problems could be converted into bi-objective mathematical programming models. Then, the bi-objective mathematical programming models were also solved using the existing methods of multi-objective programming. Cevikel and Ahlatçoglu [4] found fuzzy payoffs and fuzzy goals of two-person zero-sum games. The payoff matrix with elements was represented as a fuzzy number. For any pair of the strategies, a player received a payoff that meant as a fuzzy number. Nan *et al* [5], focused a lexicographic method for matrix games with payoffs of triangular intuitionistic fuzzy numbers. Moreover Nan *et al* [6] also introduced a new ranking method based on the value and used to solving matrix games with payoffs on trapezoidal intuitionistic fuzzy numbers (TrIFNs) fuzzy goals. Hence, there were also found the use of linear programming method for solving matrix games [7]-[8]. Aggarwal *et al* [9], solved the matrix games with I-fuzzy payoffs: Pareto-optimal security strategies approach. Therefore, in this work, two-person zero-sum games with fuzzy payoffs, matrix games with payoffs of TrIFNs were considered. The concepts of TrIFNs and their arithmetic operations were uesd. The matrix games of the cut-set based method with payoffs of TrIFNs was aimed. The auxiliary linear programming models were computed the interval-type value of any alfa-constrategies. This paper is organized as follows. In section 2, the definition, operations of TrIFNs and a methodology for matrix games with payoffs TrIFN were focused. In section 3, matrix games with payoffs of TrIFNs were formulated. In section 4, an application to voting share problem was reported. In the last section, section 5, conclusion was summerized.

2 Mathematical Preliminaries

In this section, we summarize some basic concept intuitionistic fuzzy set by Atanassov [1], notation, definition and operation of trapezoidal intuitionistic fuzzy number which are used throughout the paper.

2.1 Some definitions of TrIFNs

Definition 1. *[1] Let X be a nonempty set of the universe. If there are two mapping on the set X:*

$$\mu_{\widetilde{A}} : X \to [0,1]$$
$$x \mapsto \mu_{\widetilde{A}}(x)$$

and

$$\nu_{\widetilde{A}} : X \to [0,1]$$
$$x \mapsto \nu_{\widetilde{A}}(x)$$

with the condition $0 \leq \mu_{\widetilde{A}}(x) + \nu_{\widetilde{A}}(x) \leq 1$. The $\mu_{\widetilde{A}}$ and $\nu_{\widetilde{A}}$ are called determining and intuitionistic fuzzy set $\widetilde{A}$ on the universal set X, denote by $\{\langle x; \mu_{\widetilde{A}}(x), \nu_{\widetilde{A}}(x)\rangle | x \in X\}$ we called $\mu_{\widetilde{A}}$ and $\nu_{\widetilde{A}}$ are membership function and nonmembership function of $\widetilde{A}$, respectively. $\mu_{\widetilde{A}}(x)$ and $\nu_{\widetilde{A}}(x)$ are called the membership degree and nonmembership degree of an element x belonging to $\widetilde{A} \subseteq X$, respectively. $F(X)$ is called the set of the intuitionistic fuzzy set on the universal set X.

Definition 2. *A TrIFN $\widetilde{A} = \langle (l,c,d,r); t_{\widetilde{A}}, z_{\widetilde{A}}\rangle$ is a special intuitionistic fuzzy set on the real number set $\mathbb{R}$, whose membership and nonmembership functions are defined as follows:*

$$\mu_{\widetilde{A}}(x) = \begin{cases} 0 & \text{if } x < l \\ t_{\widetilde{A}}(x-l)/(c-l) & \text{if } l \leq x < c \\ t_{\widetilde{A}} & \text{if } c \leq x \leq d \\ t_{\widetilde{A}}(r-x)/(r-d) & \text{if } d < x \leq r \\ 0 & \text{if } x > r \end{cases} \tag{1}$$

and

$$\nu_{\widetilde{A}}(x) = \begin{cases} 1 & \text{if } x < l \\ [c - x + z_{\widetilde{A}}(x-l)]/(c-l) & \text{if } l \leq x < c \\ z_{\widetilde{A}} & \text{if } c \leq x \leq d \\ [x - d + z_{\widetilde{A}}(r-x)/(r-d)] & \text{if } d < x \leq r \\ 1 & \text{if } x > r \end{cases} \tag{2}$$

respectively, where $l \leq c \leq d \leq r$, the values $t_{\widetilde{A}}$ and $z_{\widetilde{A}}$ are maximum membership degree and minimum nonmembership degree of $\widetilde{A}$, respectively,

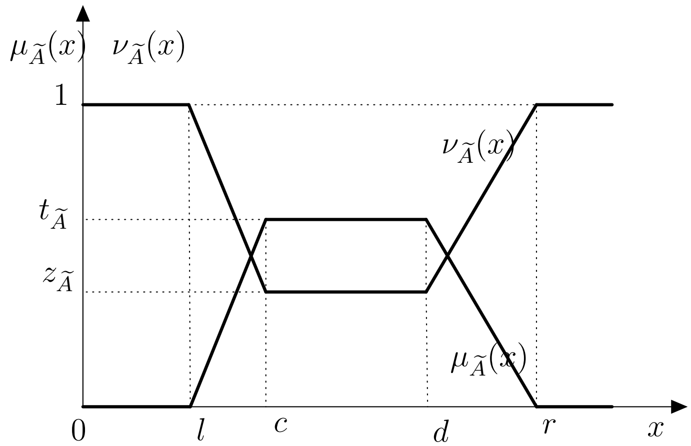

Fig. 1: A TrIFN $\widetilde{A} = \langle (l, c, d, r); t_{\widetilde{A}}, z_{\widetilde{A}} \rangle$

such that they satisfy the following condition: $t_{\widetilde{A}} \in [0,1], z_{\widetilde{A}} \in [0,1]$ *and* $t_{\widetilde{A}} + z_{\widetilde{A}} \in [0,1]$.

Let

$$\pi_{\widetilde{A}}(x) = 1 - \mu_{\widetilde{A}}(x) - \nu_{\widetilde{A}}(x) \tag{3}$$

$\pi_{\widetilde{A}}(x)$ is called the hesitancy degree of an element $x \in \widetilde{A}$. It is the degree of indeterminacy membership of the element x to $\widetilde{A}$.

From Definition 2, it is obvious that $\mu_{\widetilde{A}}(x) + \nu_{\widetilde{A}}(x) = 1$ for any $x \in \mathbb{R}$ if $t_{\widetilde{A}} = 1$ and $z_{\widetilde{A}} = 0$. Hence, the TrIFN $\widetilde{A} = \langle (l, c, d, r); t_{\widetilde{A}}, z_{\widetilde{A}} \rangle$ degenerates to $\widetilde{A} = \langle (l, c, d, r); 1, 0 \rangle$, which is a trapezoidal fuzzy number [10]. Therefore, the concept of the TrIFN is generalization of that of the trapezoidal fuzzy number.

From $\widetilde{A} = \langle (l, c, d, r); t_{\widetilde{A}}, z_{\widetilde{A}} \rangle$ if $c = d = p$ then $\widetilde{A} = \langle (l, p, r); t_{\widetilde{A}}, z_{\widetilde{A}} \rangle$ that is $\widetilde{A} = \langle (l, p, r); t_{\widetilde{A}}, z_{\widetilde{A}} \rangle$ is a triangular intuitionistic fuzzy number (TIFN), which is particular case of TrIFN. Likewise to algebraic operations of TIFN and TrIFN are defined as follows.

Definition 3. *Let* $\widetilde{A} = \langle (l_1, c_1, d_1, r_1); t_{\widetilde{A}}, z_{\widetilde{A}} \rangle$ *and* $\widetilde{B} = \langle (l_2, c_2, d_2, r_2); t_{\widetilde{B}}, z_{\widetilde{B}} \rangle$ *be two TrIFNs with* $t_{\widetilde{A}} \neq t_{\widetilde{B}}$ *and* $z_{\widetilde{A}} \neq z_{\widetilde{B}}$, $\gamma \neq 0$ *be any real number. Then, the algebraic operations of TrIFNs are defined as follows:*

$$\widetilde{A} + \widetilde{B} = \langle (l_1 + l_2, c_1 + c_2, d_1 + d_2, r_1 + r_2); t_{\widetilde{A}} \wedge t_{\widetilde{B}}, z_{\widetilde{A}} \vee z_{\widetilde{B}} \rangle, \tag{4}$$

$$\widetilde{A} - \widetilde{B} = \langle (l_1 - r_2, c_1 - d_2, d_1 - c_2, r_1 - l_2); t_{\widetilde{A}} \wedge t_{\widetilde{B}}, z_{\widetilde{A}} \vee z_{\widetilde{B}} \rangle, \tag{5}$$

$$\widetilde{A}\widetilde{B} = \begin{cases} \langle (l_1 l_2, c_1 c_2, d_1 d_2, r_1 r_2); t_{\widetilde{A}} \wedge t_{\widetilde{B}}, z_{\widetilde{A}} \vee z_{\widetilde{B}} \rangle & \textit{if } \widetilde{A} > 0, \widetilde{B} > 0 \\ \langle (l_1 r_2, c_1 d_2, d_1 c_2, r_1 l_2); t_{\widetilde{A}} \wedge t_{\widetilde{B}}, z_{\widetilde{A}} \vee z_{\widetilde{B}} \rangle & \textit{if } \widetilde{A} < 0, \widetilde{B} > 0 \\ \langle (r_1 r_2, d_1 d_2, c_1 c_2, l_1 l_2); t_{\widetilde{A}} \wedge t_{\widetilde{B}}, z_{\widetilde{A}} \vee z_{\widetilde{B}} \rangle & \textit{if } \widetilde{A} < 0, \widetilde{B} < 0, \end{cases} \tag{6}$$

$$\widetilde{A}/\widetilde{B} = \begin{cases} \langle (l_1/r_2, c_1/d_2, d_1/c_2, r_1/l_2); t_{\widetilde{A}} \wedge t_{\widetilde{B}}, z_{\widetilde{A}} \vee z_{\widetilde{B}} \rangle & \textit{if } \widetilde{A} > 0, \widetilde{B} > 0 \\ \langle (r_1/r_2, d_1/d_2, c_1/c_2, l_1/l_2); t_{\widetilde{A}} \wedge t_{\widetilde{B}}, z_{\widetilde{A}} \vee z_{\widetilde{B}} \rangle & \textit{if } \widetilde{A} < 0, \widetilde{B} > 0 \\ \langle (r_1/l_2, d_1/c_2, c_1/d_2, l_1/r_2); t_{\widetilde{A}} \wedge t_{\widetilde{B}}, z_{\widetilde{A}} \vee z_{\widetilde{B}} \rangle & \textit{if } \widetilde{A} < 0, \widetilde{B} < 0, \end{cases} \tag{7}$$

$$\gamma/\widetilde{A} = \begin{cases} \langle (\gamma l_1, \gamma c_1, \gamma d_1, \gamma r_1); t_{\widetilde{A}}, z_{\widetilde{A}} \rangle & \textit{if } \gamma > 0 \\ \langle (\gamma r_1, \gamma d_1, \gamma c_1, \gamma l_1); t_{\widetilde{A}}, z_{\widetilde{A}} \rangle & \textit{if } \gamma < 0 \end{cases} \tag{8}$$

and

$$\widetilde{A}^{-1} = \langle (1/r_1, 1/d_1, 1/c_1, 1/l_1); t_{\widetilde{A}}, z_{\widetilde{A}} \rangle \quad \textit{if} \widetilde{A} \neq 0 \tag{9}$$

where the symbols $\wedge$ is the minimum operator and $\vee$ is the maximum operator.

2.2 A methodology for matrix games with payoffs TrIFN

Let $S_1 = \{\rho_1, \rho_2, \ldots, \rho_m\}$ and $S_2 = \{\tau_1, \tau_2, \ldots \tau_n\}$ be sets of pure strategies for players I and II, respectively. The vector $x = (x_1, x_2, \ldots, x_m)^T$ is mixed strategies for player I where $x_i\,(i = 1, 2, \ldots, m)$ is probability in player I. The set of mixed strategies for player I is represented by $X = \{x| \sum_{i=1}^{m} x_i = 1, x_i \geq 0\,(i = 1, 2, \ldots, m)\}$.

Similarly, the vector $y = (y_1, y_2, \ldots, y_n)^T$ is mixed strategies for player II where $y_j\,(j = 1, 2, \ldots, n)$ is probability in player II. The set of mixed strategies for player II is $Y = \{y| \sum_{j=1}^{n} y_j = 1, y_j \geq 0\,(j = 1, 2, \ldots, n)\}$. Assume that the payoff of players I is expressed with an TrIFN

$$\widetilde{D}_{ij} = \langle (l_{ij}, c_{ij}, d_{ij}, r_{ij}); t_{\widetilde{D}_{ij}}, z_{\widetilde{D}_{ij}} \rangle,$$

where $l_{ij} \leq c_{ij} \leq d_{ij} \leq r_{ij}$, $t_{\widetilde{D}_{ij}} \in [0,1]$ and $z_{\widetilde{D}_{ij}} \in [0,1] (i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$. Therefore, the payoff player I at all $m \times n$ pure strategy situations can be concisely expressed in the matrix format as follows:

$$\widetilde{D} = \begin{bmatrix} \widetilde{D}_{11} & \widetilde{D}_{12} & \cdots & \widetilde{D}_{1n} \\ \widetilde{D}_{21} & \widetilde{D}_{22} & \cdots & \widetilde{D}_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \widetilde{D}_{m1} & \widetilde{D}_{m2} & \cdots & \widetilde{D}_{mn} \end{bmatrix}$$

denote by $\widetilde{D} = (\langle l_{ij}, c_{ij}, d_{ij}, r_{ij} \rangle)_{m \times n}$ or $\widetilde{D} = (\widetilde{D}_{ij})_{m \times n}$.

From $\widetilde{D}_{ij} = \langle (l_{ij}, c_{ij}, d_{ij}, r_{ij}); t_{\widetilde{D}_{ij}}, z_{\widetilde{D}_{ij}} \rangle$ if $c_{ij} = d_{ij} = p_{ij} (i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$ is reduced to $\widetilde{D}_{ij} = \langle (l_{ij}, p_{ij}, r_{ij}); t_{\widetilde{D}_{ij}}, z_{\widetilde{D}_{ij}} \rangle$ such that $\widetilde{D}_{ij} = \langle (l_{ij}, p_{ij}, r_{ij});$
$t_{\widetilde{D}_{ij}}, z_{\widetilde{D}_{ij}} \rangle$ is call matrix games with payffs of TIFN, which is particular case of matrix games with payoffs of TrIFN.

Definition 4. *[11] Let* $\widetilde{\nu} = \langle (\nu_1, \nu_2, \nu_3, \nu_4); t_{\widetilde{\nu}}, z_{\widetilde{\nu}} \rangle$ *and* $\widetilde{\omega} = \langle (\omega_1, \omega_2, \omega_3, \omega_4); t_{\widetilde{\omega}}, z_{\widetilde{\omega}} \rangle$ *be TrIFNs. If there are mixed strategies* $x^* \in X$ *and* $y^* \in Y$ *so that for any mixed strategies* $x \in X$ *and* $y \in Y$ *they satisfy the two conditions as follows:*

(i) $x^{*T} \widetilde{D} y \widetilde{\geq} \widetilde{\nu}$ *and*

(ii) $x^T \widetilde{D} y^* \widetilde{\leq} \widetilde{\omega}$, *then* $(x^*, y^*, \widetilde{\nu}, \widetilde{\omega})$ *is called a reasonable solution of the matrix game* $\widetilde{D}$ *with payoffs of TrIFN.* x^* *and* y^* *are called reasonable strategy for player I and II, respectively.* $\widetilde{\nu}$ *and* $\widetilde{\omega}$ *are called reasonable values of players I and II, respectively.*

The symbol "$\widetilde{\leq}$", "$\widetilde{\geq}$" and "$\widetilde{=}$" are an intuitionistic fuzzy version of the order relation "$\leq$", "$\geq$" and "$=$" on the real number set. The sets of all reasonable values $\widetilde{\nu}$ and $\widetilde{\omega}$ for players I and II are denoted by V and W, respectively.

Definition 5. *[11] Assume that there exist reasonable values* $\widetilde{\nu}^* \in V$ *and* $\widetilde{\omega}^* \in W$ *for playersI and II, respectively. If there do not exist any reasonable values* $\widetilde{\nu} \in V (\widetilde{\nu} \neq \widetilde{\nu}^*)$ *and* $\widetilde{\omega} \in W (\widetilde{\omega} \neq \widetilde{\omega}^*)$ *so that they satisfy the conditions as follows:*

(i) $\widetilde{\nu}\tilde{\geq}\widetilde{\nu}^*$ *and,*

(ii) $\widetilde{\omega}\tilde{\leq}\widetilde{\omega}^*$, *then* $(x^*, y^*, \widetilde{\nu}^*, \widetilde{\omega}^*)$ *is called a solution of the matrix game* $\widetilde{D}$ *with payoffs of trapezoidal intuitionistic fuzzy numbers.* x^* *and* y^* *are called the maximin strategy and minimax strategy for players I and II, respectively.* $\widetilde{\nu}^*$ *and* $\widetilde{\omega}^*$ *are called the gain-floor of player I and the loss-ceiling of player II, respectively.* ${x^*}^T\widetilde{D}y^*$ *is called the value of the matrix game* $\widetilde{D}$ *with payoffs of TrIFN.*

2.3 Cut sets of TrIFN

Definition 6. *[5] A* (α, λ)*- cut set of* $\widetilde{A} = \langle(l, c, d, r); t_{\widetilde{A}}, z_{\widetilde{A}}\rangle$ *is a crisp subset of* $\mathbb{R}$*, which is defined as follows:*

$$\widetilde{A}_\alpha^\lambda = \{x|\mu_{\widetilde{A}}(x) \geq \alpha, \nu_{\widetilde{A}}(x) \leq \lambda\},$$

where $0 \leq \alpha \leq t_{\widetilde{A}}, z_{\widetilde{A}} \leq \lambda \leq 1$ *and* $0 \leq \alpha + \lambda \leq 1$.

Definition 7. *[5] The* α*-cut set and* λ*-cut set of* $\widetilde{A} = \langle(l, c, d, r); t_{\widetilde{A}}, z_{\widetilde{A}}\rangle$ *are a crisp subset of* $\mathbb{R}$*, which is defined as follows:*

$$\widetilde{A}_\alpha = \{x|\mu_{\widetilde{A}}(x) \geq \alpha\}$$

and

$$\widetilde{A}^\lambda = \{x|\nu_{\widetilde{A}}(x) \leq \lambda\}$$

respectively.

Using the membership function of $\widetilde{A} = \langle(l, c, d, r); t_{\widetilde{A}}, z_{\widetilde{A}}\rangle$ and Definition 7 such that $\widetilde{A}_\alpha = \{x|\mu_{\widetilde{A}}(x) \geq \alpha\}$ and $\widetilde{A}^\lambda = \{x|\nu_{\widetilde{A}}(x) \leq \lambda\}$ are closed interval and calculated as follows:

$$\widetilde{A}_\alpha = \left[L_{\widetilde{A}}(\alpha), R_{\widetilde{A}}(\alpha)\right] = \left[\frac{(t_{\widetilde{A}} - \alpha)l + \alpha c}{t_{\widetilde{A}}}, \frac{(t_{\widetilde{A}} - \alpha)r + \alpha d}{t_{\widetilde{A}}}\right] \tag{10}$$

and

$$\widetilde{A}^\lambda = \left[L'_{\widetilde{A}}(\lambda), R'_{\widetilde{A}}(\lambda)\right] = \left[\frac{(1-\lambda)c + (\lambda - z_{\widetilde{A}})l}{1 - z_{\widetilde{A}}}, \frac{(1-\lambda)d + (\lambda - z_{\widetilde{A}})r}{1 - z_{\widetilde{A}}}\right] \tag{11}$$

respectively.

Definition 8. *[12] Assume that the α-cut set and λ-cut set of any TrIFNs $\widetilde{A}$ and $\widetilde{B}$ are $\widetilde{A}_\alpha = [L_{\widetilde{A}}(\alpha), R_{\widetilde{A}}(\alpha)]$, $\widetilde{A}^\lambda = [L'_{\widetilde{A}}(\lambda), R'_{\widetilde{A}}(\lambda)]$, $\widetilde{B}_\alpha = [L_{\widetilde{B}}(\alpha), R_{\widetilde{B}}(\alpha)]$ and $\widetilde{B}^\lambda = [L'_{\widetilde{B}}(\lambda), R'_{\widetilde{B}}(\lambda)]$, respectively. Then the ranking order of the TrIFNs $\widetilde{A}$ and $\widetilde{B}$ is stipulated according to the two cases as follws:*

(i) If $L_{\widetilde{B}}(\alpha) \geq L_{\widetilde{A}}(\alpha)$, $R_{\widetilde{B}}(\alpha) \geq R_{\widetilde{A}}(\alpha)$, $L'_{\widetilde{B}}(\lambda) \geq L'_{\widetilde{A}}(\lambda)$ and $R'_{\widetilde{B}}(\lambda) \geq R'_{\widetilde{A}}(\lambda)$, then $\widetilde{B} \geq \widetilde{A}$

(ii) If $L_{\widetilde{B}}(\alpha) \leq L_{\widetilde{A}}(\alpha)$, $R_{\widetilde{B}}(\alpha) \leq R_{\widetilde{A}}(\alpha)$, $L'_{\widetilde{B}}(\lambda) \leq L'_{\widetilde{A}}(\lambda)$ and $R'_{\widetilde{B}}(\lambda) \leq R'_{\widetilde{A}}(\lambda)$, then $\widetilde{B} \leq \widetilde{A}$.

3 Mathematical programming model for the TrIFN matrix game

As the TrIFN matrix game is a zero - sum game, from Definition 3 expected payoff for player I is computed as follows:

$$\begin{aligned}
E(x, y) &= x^T \widetilde{D} y \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \widetilde{D}_{ij} x_i y_j \\
&= \langle (\sum_{i=1}^{m} \sum_{j=1}^{n} l_{ij} x_i y_j, \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_i y_j, \sum_{i=1}^{m} \sum_{j=1}^{n} d_{ij} x_i y_j, \sum_{i=1}^{m} \sum_{j=1}^{n} r_{ij} x_i y_j); \\
&\wedge \{t_{\widetilde{D}_{ij}}\}, \vee \{z_{\widetilde{D}_{ij}}\} \rangle
\end{aligned}$$

which is a TrIFN.

And the expected payoff for player II obtained as follows:

$$\begin{aligned}
E(x, y) &= x^T (-\widetilde{D}) y \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} (-\widetilde{D}_{ij}) x_i y_j \\
&= \langle (-\sum_{i=1}^{m} \sum_{j=1}^{n} r_{ij} x_i y_j, -\sum_{i=1}^{m} \sum_{j=1}^{n} d_{ij} x_i y_j, -\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_i y_j, \\
&- \sum_{i=1}^{m} \sum_{j=1}^{n} l_{ij} x_i y_j); \wedge \{t_{\widetilde{D}_{ij}}\}, \vee \{z_{\widetilde{D}_{ij}}\} \rangle.
\end{aligned}$$

Assume that player I is a maximizing player and player II is a minimizing player. Player I should choose a mixed strategy $x \in X$ that maximizes the minimum expected gain of player II, i.e.,

$$\widetilde{\nu} = \max_{x\in X} \min_{y\in Y}\{E(x,y)\}, \tag{12}$$

which is called player I's gain-floor.

Similarly, player II should choose a mixed strategy $y \in Y$ that minimizes the maximum expected loss of player I, i.e.,

$$\widetilde{\omega} = \min_{y\in Y} \max_{x\in X}\{E(x,y)\}, \tag{13}$$

which is called player II's loss-ceiling.

Hence, player I's gain-floor and player II's loss ceiling denoted by

$$\widetilde{\nu} = \langle(\nu_1,\nu_2,\nu_3,\nu_4); t_{\widetilde{\nu}}, z_{\widetilde{\nu}}\rangle$$

and

$$\widetilde{\omega} = \langle(\omega_1,\omega_2,\omega_3,\omega_4); t_{\widetilde{\omega}}, z_{\widetilde{\omega}}\rangle.$$

From Definitions 4, 5 and Eqs.(12) and (13) the maximin strategy x^* and gain-floor $\widetilde{\nu}^*$ of player I and the minimax strategy y^* and loss-ceiling $\widetilde{\omega}^*$ of player II can be generate by solving an intuitionistic fuzzy mathematical programming model constructed as follows:

$$\max\{\widetilde{\nu}\}$$
$$s.t. \begin{cases} \sum_{i=1}^{m} \widetilde{D}_{ij} x_i y_j \widetilde{\geq} \widetilde{\nu} \quad (j = 1,2,\ldots,n)(y \in Y) \\ \sum_{i=1}^{m} x_i = 1 \\ x_i \geq 0 \quad (i = 1,2,\ldots,m) \end{cases} \tag{14}$$

and

$$\min\{\widetilde{\omega}\}$$
$$s.t. \begin{cases} \sum_{j=1}^{n} \widetilde{D}_{ij} x_i y_j \widetilde{\leq} \widetilde{\omega} \quad (i = 1,2,\ldots,m)(x \in X) \\ \sum_{i=1}^{n} y_j = 1 \\ y_j \geq 0 \quad (j = 1,2,\ldots,n), \end{cases} \tag{15}$$

respectively, where $\widetilde{\nu}$ and $\widetilde{\omega}$ are TrIFNs.

From Eqs.(14), (15) and theorem [6] can be converted into an intuitionistic fuzzy mathematical programming models as follows:

$$\max\{\widetilde{\nu}\}$$

$$s.t.\begin{cases}\sum_{i=1}^{m}\widetilde{D}_{ij}x_i\widetilde{\geq}\widetilde{\nu} \quad (j=1,2,\ldots,n)\\ \sum_{i=1}^{m}x_i=1\\ x_i\geq 0 \quad (i=1,2,\ldots,m)\end{cases} \tag{16}$$

and

$$\min\{\widetilde{\omega}\}$$

$$s.t.\begin{cases}\sum_{j=1}^{n}\widetilde{D}_{ij}y_j\widetilde{\leq}\widetilde{\omega} \quad (i=1,2,\ldots,m)\\ \sum_{j=1}^{n}y_j=1\\ y_j\geq 0 \quad (j=1,2,\ldots,n)\end{cases} \tag{17}$$

respectively. From Definitions 6 and 8 can be transformed into the interval-valued bi-objective mathematical programming models as follows:

$$\max\{\widetilde{\nu}_\alpha,\widetilde{\nu}^\lambda\}$$

$$s.t.\begin{cases}\sum_{i=1}^{m}(\widetilde{D}_{ij})_\alpha x_i\geq\widetilde{\nu}_\alpha \quad (j=1,2,\ldots,n)\\ \sum_{i=1}^{m}(\widetilde{D}_{ij})^\lambda x_i\geq\widetilde{\nu}^\lambda \quad (j=1,2,\ldots,n)\\ \sum_{i=1}^{m}x_i=1\\ x_i\geq 0 \quad (i=1,2,\ldots,m)\end{cases} \tag{18}$$

the α-cut set and λ-cut set of the TrIFNs $\widetilde{\nu}$ and $\widetilde{D}_{ij}=(i=1,2,\ldots,m;j=1,2,\ldots,n)$ are denoted by $\widetilde{\nu}_\alpha=[\nu_\alpha^L,\nu_\alpha^R],\widetilde{\nu}^\lambda=[\nu_L^\lambda,\nu_R^\lambda]$, $(\widetilde{D}_{ij})_\alpha=[L_{\widetilde{D}_{ij}}(\alpha),R_{\widetilde{D}_{ij}}(\alpha)]$

and $(\widetilde{D}_{ij})^{\lambda} = [L'_{\widetilde{D}_{ij}}(\lambda), R'_{\widetilde{D}_{ij}}(\lambda)]$, respectively. From Eq.(18) can be written as the following interval-valued bi-object mathematical programming model:

$$\max\{[\nu_{\alpha}^{L}, \nu_{\alpha}^{R}], [\nu_{L}^{\lambda}, \nu_{R}^{\lambda}]\}$$

$$s.t. \begin{cases} \sum_{i=1}^{m} L_{\widetilde{D}_{ij}}(\alpha)x_i \geq \nu_{\alpha}^{L} \quad (j = 1, 2, \dots, n) \\ \sum_{i=1}^{m} R_{\widetilde{D}_{ij}}(\alpha)x_i \geq \nu_{\alpha}^{R} \quad (j = 1, 2, \dots, n) \\ \sum_{i=1}^{m} L'_{\widetilde{D}_{ij}}(\lambda)x_i \geq \nu_{L}^{\lambda} \quad (j = 1, 2, \dots, n) \\ \sum_{i=1}^{m} R'_{\widetilde{D}_{ij}}(\lambda)x_i \geq \nu_{R}^{\lambda} \quad (j = 1, 2, \dots, n) \\ \sum_{i=1}^{m} x_i = 1 \\ x_i \geq 0 \quad (i = 1, 2, \dots, m) \end{cases} \tag{19}$$

where $\nu_{\alpha}^{L}, \nu_{\alpha}^{R}, \nu_{L}^{\lambda}, \nu_{R}^{\lambda}$ and $x_i (i = 1, 2, \dots, m)$ are decision variables.

In Eq.(19) two interval-valued objective functions, we will using the linear weighted averaging method of multiobjective decision making, their weights are the same as $1/2$ hence, Eq.(19) can be aggregated into the interval-valued mathematical programming model as follows:

$$\max\left\{\left[\frac{\nu_\alpha^L+\nu_L^\lambda}{2},\frac{\nu_\alpha^R+\nu_R^\lambda}{2}\right]\right\}$$

$$s.t.\begin{cases}\sum_{i=1}^{m} L_{\widetilde{D}_{ij}}(\alpha)x_i \geq \nu_\alpha^L \quad (j=1,2,\dots,n)\\ \sum_{i=1}^{m} R_{\widetilde{D}_{ij}}(\alpha)x_i \geq \nu_\alpha^R \quad (j=1,2,\dots,n)\\ \sum_{i=1}^{m} L'_{\widetilde{D}_{ij}}(\lambda)x_i \geq \nu_L^\lambda \quad (j=1,2,\dots,n)\\ \sum_{i=1}^{m} R'_{\widetilde{D}_{ij}}(\lambda)x_i \geq \nu_R^\lambda \quad (j=1,2,\dots,n)\\ \sum_{i=1}^{m} x_i = 1\\ x_i \geq 0 \quad (i=1,2,\dots,m).\end{cases} \tag{20}$$

From Eq.(20) and Ishibuchi and Ianaka [2] the maximization problem with the interval- valued objective function can be written as follows:

$$\max\left\{\frac{\nu_\alpha^L+\nu_L^\lambda}{2},\frac{\nu_\alpha^L+\nu_L^\lambda+\nu_\alpha^R+\nu_R^\lambda}{4}\right\}$$

$$s.t.\begin{cases}\sum_{i=1}^{m} L_{\widetilde{D}_{ij}}(\alpha)x_i \geq \nu_\alpha^L \quad (j=1,2,\dots,n)\\ \sum_{i=1}^{m} R_{\widetilde{D}_{ij}}(\alpha)x_i \geq \nu_\alpha^R \quad (j=1,2,\dots,n)\\ \sum_{i=1}^{m} L'_{\widetilde{D}_{ij}}(\lambda)x_i \geq \nu_L^\lambda \quad (j=1,2,\dots,n)\\ \sum_{i=1}^{m} R'_{\widetilde{D}_{ij}}(\lambda)x_i \geq \nu_R^\lambda \quad (j=1,2,\dots,n)\\ \sum_{i=1}^{m} x_i = 1\\ x_i \geq 0 \quad (i=1,2,\dots,m).\end{cases} \tag{21}$$

From Eq.(21) and using the linear weighted averaging method of multi-objective decision making [13] - [14] can be written as follows:

$$\max\left\{\xi\frac{\nu_\alpha^L+\nu_L^\lambda}{2}+(1-\xi)\frac{\nu_\alpha^L+\nu_L^\lambda+\nu_\alpha^R+\nu_R^\lambda}{4}\right\}$$

$$s.t.\begin{cases}\sum_{i=1}^{m} L_{\widetilde{D}_{ij}}(\alpha)x_i \geq \nu_\alpha^L \quad (j=1,2,\ldots,n)\\ \sum_{i=1}^{m} R_{\widetilde{D}_{ij}}(\alpha)x_i \geq \nu_\alpha^R \quad (j=1,2,\ldots,n)\\ \sum_{i=1}^{m} L'_{\widetilde{D}_{ij}}(\lambda)x_i \geq \nu_L^\lambda \quad (j=1,2,\ldots,n)\\ \sum_{i=1}^{m} R'_{\widetilde{D}_{ij}}(\lambda)x_i \geq \nu_R^\lambda \quad (j=1,2,\ldots,n)\\ \sum_{i=1}^{m} x_i = 1\\ x_i \geq 0 \quad (i=1,2,\ldots,m)\end{cases} \tag{22}$$

where $\xi \in [0,1]$.

From Eqs.(10) and (11) we can obtain the α-cut set and λ-cut set of the TrIFNs $\widetilde{D}_{ij} = \langle (l_{ij}, c_{ij}, d_{ij}, r_{ij}); t_{\widetilde{D}_{ij}}, z_{\widetilde{D}_{ij}}\rangle (i = 1,2,\ldots,m; j = 1,2,\ldots,n)$ as follows:

$$(\widetilde{D}_{ij})_\alpha = \left[L_{\widetilde{D}_{ij}}(\alpha), R_{\widetilde{D}_{ij}}(\alpha)\right] = \left[\frac{(t_{\widetilde{D}_{ij}}-\alpha)l_{ij}+\alpha c_{ij}}{t_{\widetilde{D}_{ij}}}, \frac{(t_{\widetilde{D}_{ij}}-\alpha)r_{ij}+\alpha d_{ij}}{t_{\widetilde{D}_{ij}}}\right] \tag{23}$$

and

$$(\widetilde{D}_{ij})^\lambda = \left[L'_{\widetilde{D}_{ij}}(\lambda), R'_{\widetilde{D}_{ij}}(\lambda)\right] = \left[\frac{(1-\lambda)a_{ij}+(\lambda-z_{\widetilde{D}_{ij}})l_{ij}}{1-z_{\widetilde{D}_{ij}}}, \frac{(1-\lambda)d_{ij}+(\lambda-u_{\widetilde{D}_{ij}})r_{ij}}{1-z_{\widetilde{D}_{ij}}}\right] \tag{24}$$

respectively.

From Eq.(22) can rewritten as the following linear programming model:

$$
\max\left\{\xi\frac{\nu_\alpha^L+\nu_L^\lambda}{2}+(1-\xi)\frac{\nu_\alpha^L+\nu_L^\lambda+\nu_\alpha^R+\nu_R^\lambda}{4}\right\}
$$

$$
s.t.\begin{cases}
\sum_{i=1}^{m}\dfrac{(t_{\widetilde{D}_{ij}}-\alpha)l_{ij}+\alpha c_{ij}}{t_{\widetilde{D}_{ij}}}x_i\geq\nu_\alpha^L\quad(j=1,2,\ldots,n)\\
\sum_{i=1}^{m}\dfrac{(t_{\widetilde{D}_{ij}}-\alpha)r_{ij}+\alpha d_{ij}}{t_{\widetilde{D}_{ij}}}x_i\geq\nu_\alpha^R\quad(j=1,2,\ldots,n)\\
\sum_{i=1}^{m}\dfrac{(1-\lambda)c_{ij}+(\lambda-z_{\widetilde{D}_{ij}})l_{ij}}{1-z_{\widetilde{D}_{ij}}}x_i\geq\nu_L^\lambda\quad(j=1,2,\ldots,n)\\
\sum_{i=1}^{m}\dfrac{(1-\lambda)d_{ij}+(\lambda-z_{\widetilde{D}_{ij}})r_{ij}}{1-z_{\widetilde{D}_{ij}}}x_i\geq\nu_R^\lambda\quad(j=1,2,\ldots,n)\\
\sum_{i=1}^{m}x_i=1\\
x_i\geq 0\quad(i=1,2,\ldots,m)
\end{cases}\tag{25}
$$

where $\xi\in[0,1]$, $\alpha\in[0,t_{\widetilde{D}}]$, $\lambda\in[z_{\widetilde{D}},1]$ and $\alpha+\lambda\in[0,1]$.

Using the simplex method of linear programming, we can obtain the optimal solution of Eq.(25), denote by $(x^*(\alpha,\lambda),\nu_\alpha^{L*},\nu_\alpha^{R*},\nu_L^{\lambda*},\nu_R^{\lambda*})$. where $x^*(\alpha,\lambda)$ is the maximin strategy of player I at the $\langle\alpha,\lambda\rangle$-confidence level. ν_α^{L*} and ν_α^{R*} are the lower and upper bounds of the gain-floor $\widetilde{\nu}^*$ of player I at the α-confidence level, that is α- cut set $\widetilde{\nu}_\alpha^*$ of $\widetilde{\nu}^*$. Similarly, $\nu_L^{\lambda*}$ and $\nu_R^{\lambda*}$ are the lower and upper bounds of the gain-floor $\widetilde{\nu}^*$ of player I at the λ-confidence level, that is that is λ-cut set $\widetilde{\nu}_\lambda^*$ of $\widetilde{\nu}^*$.

In the same way, from Defintion 8 and Eq.(17) can be written as the following interval-valued bi-objective mathematical programming model:

$$
\min\{\widetilde{\omega}_\alpha,\widetilde{\omega}^\lambda\}
$$

$$
s.t.\begin{cases}
\sum_{j=1}^{n}(\widetilde{D}_{ij})_\alpha y_j\leq\widetilde{\omega}_\alpha\quad(i=1,2,\ldots,m)\\
\sum_{j=1}^{n}(\widetilde{D}_{ij})^\lambda y_j\leq\widetilde{\omega}^\lambda\quad(i=1,2,\ldots,m)\\
\sum_{j=1}^{n}y_j=1\\
y_j\geq 0\quad(j=1,2,\ldots,n).
\end{cases}\tag{26}
$$

The α-cut set and λ-cut set of the TrIFN $\widetilde{\omega}$ are denote by $\widetilde{\omega}_\alpha = \left[\omega_\alpha^L, \omega_\alpha^R\right]$ and $\widetilde{\omega}^\lambda = \left[\omega_L^\lambda, \omega_R^\lambda\right]$, respectively. From Eq.[26] can be written as the folowing interval-valued bi-objective mathematical programming model:

$$\min\{[\omega_\alpha^L, \omega_\alpha^R], [\omega_L^\lambda, \omega_R^\lambda]\}$$

$$s.t. \begin{cases} \sum_{i=1}^{n} L_{\widetilde{D}_{ij}}(\alpha) y_j \leq \omega_R^L \quad (i = 1, 2, \ldots, m) \\ \sum_{i=1}^{n} R_{\widetilde{D}_{ij}}(\alpha) y_j \leq \omega_\alpha^R \quad (i = 1, 2, \ldots, m) \\ \sum_{i=1}^{n} L'_{\widetilde{D}_{ij}}(\lambda) y_j \leq \omega_L^\lambda \quad (i = 1, 2, \ldots, m) \\ \sum_{i=1}^{n} R'_{\widetilde{D}_{ij}}(\lambda) y_j \leq \omega_R^\lambda \quad (i = 1, 2, \ldots, m) \\ \sum_{i=1}^{n} y_j = 1 \\ y_j \geq 0 \quad (j = 1, 2, \ldots, n) \end{cases} \tag{27}$$

where $\omega_\alpha^L, \omega_\alpha^R, \omega_L^\lambda, \omega_R^\lambda$ and $y_j\,(i = 1, 2, \ldots, n)$ are decision variables.

From Eq.(27) and use the linear weighted veraging method of multiobjective decisionmaking [13] - [14] can be aggregated into the interval-valued mathematical programming model as follows:

$$
\min\left\{\left[\frac{\omega_\alpha^L+\omega_L^\lambda}{2},\frac{\omega_\alpha^R+\omega_R^\lambda}{2}\right]\right\}
$$

$$
s.t.\begin{cases}\sum_{i=1}^{n} L_{\widetilde{D}_{ij}}(\alpha)y_j \le \omega_R^L \quad (i=1,2,\ldots,m)\\ \sum_{i=1}^{n} R_{\widetilde{D}_{ij}}(\alpha)y_j \le \omega_\alpha^R \quad (i=1,2,\ldots,m)\\ \sum_{i=1}^{n} L'_{\widetilde{D}_{ij}}(\lambda)y_j \le \omega_L^\lambda \quad (i=1,2,\ldots,m)\\ \sum_{i=1}^{n} R'_{\widetilde{D}_{ij}}(\lambda)y_j \le \omega_R^\lambda \quad (i=1,2,\ldots,m)\\ \sum_{i=1}^{m} y_j=1\\ y_j\ge 0 \quad (i=1,2,\ldots,n).\end{cases} \tag{28}
$$

From Eq.(28) and the maximization problem with the interval-valued objective function [2] can be written as follows:

$$
\min\left\{\frac{\omega_\alpha^R+\omega_R^\lambda}{2}+\frac{\omega_\alpha^L+\omega_L^\lambda+\omega_\alpha^R+\omega_R^\lambda}{4}\right\}
$$

$$
s.t.\begin{cases}\sum_{i=1}^{n} L_{\widetilde{D}_{ij}}(\alpha)y_j \le \omega_\alpha^L \quad (i=1,2,\ldots,m)\\ \sum_{i=1}^{n} R_{\widetilde{D}_{ij}}(\alpha)y_j \le \omega_\alpha^R \quad (i=1,2,\ldots,m)\\ \sum_{i=1}^{n} L'_{\widetilde{D}_{ij}}(\lambda)y_j \le \omega_L^\lambda \quad (i=1,2,\ldots,m)\\ \sum_{i=1}^{n} R'_{\widetilde{D}_{ij}}(\lambda)y_j \le \omega_R^\lambda \quad (i=1,2,\ldots,m)\\ \sum_{i=1}^{n} y_j=1\\ y_j\ge 0 \quad (j=1,2,\ldots,n).\end{cases} \tag{29}
$$

Similary, using the linear weighted averaging method of multiobjective decision making, from Eqs.(23), (24) and (29) can be further aggregated and convert into the linear programming model as follows:

$$\min\left\{\xi\frac{\omega_\alpha^R+\omega_R^\lambda}{2}+(1-\xi)\frac{\omega_\alpha^L+\omega_L^\lambda+\omega_\alpha^R+\omega_R^\lambda}{4}\right\}$$

$$s.t.\begin{cases}\sum_{j=1}^{n}\frac{(t_{\widetilde{D}_{ij}}-\alpha)l_{ij}+\alpha c_{ij}}{t_{\widetilde{D}_{ij}}}y_j\le\omega_\alpha^L\quad(i=1,2,\ldots,m)\\ \sum_{j=1}^{n}\frac{(t_{\widetilde{D}_{ij}}-\alpha)r_{ij}+\alpha d_{ij}}{t_{\widetilde{D}_{ij}}}y_j\le\omega_\alpha^R\quad(i=1,2,\ldots,m)\\ \sum_{j=1}^{n}\frac{(1-\lambda)c_{ij}+(\lambda-z_{\widetilde{D}_{ij}})l_{ij}}{1-z_{\widetilde{D}_{ij}}}y_i\le\omega_L^\lambda\quad(i=1,2,\ldots,m)\\ \sum_{j=1}^{n}\frac{(1-\lambda)d_{ij}+(\lambda-z_{\widetilde{D}_{ij}})r_{ij}}{1-z_{\widetilde{D}_{ij}}}y_i\le\omega_R^\lambda\quad(i=1,2,\ldots,m)\\ \sum_{j=1}^{n}y_j=1\\ y_j\ge 0\quad(j=1,2,\ldots,n).\end{cases}\tag{30}$$

For any adequately given values of the parameters ξ,α and λ, using the simplex medthod of linear programming, we can obtain the optimal solution of Eq.(30), denote by $(y^*(\alpha,\lambda),\omega_\alpha^{L*},\omega_\alpha^{R*},\omega_L^{\lambda*},\omega_R^{\lambda*})$ where $y^*(\alpha,\lambda)$ is the minimax strategy of player II at the $\langle\alpha,\lambda\rangle$-confidence level. ω_α^{L*} and ω_α^{R*} are the lower and upper bounds of the gain-floor $\widetilde{\omega}^*$ of player II at the α-confidence level, that is α- cut set $\widetilde{\omega}_\alpha^*$ of $\widetilde{\omega}^*$. similarly, $\omega_L^{\lambda*}$ and $\omega_R^{\lambda*}$ are the lower and upper bounds of the gain-floor $\widetilde{\omega}^*$ of player II at the λ-confidence level, that is that is λ- cut set $\widetilde{\omega}_\lambda^*$ of $\widetilde{\omega}^*$.

4 An application to voting share problem

In this examples 1 and 2, using Eqs.(25) and (30) to solve the problem.

Example 1.

Assume that there is an election where two major political parties M and W participate and total number of voters in that regions is stable. It means that the increase in percentage of votes for one political party results in the same for the other political party. Suppose M has two strategies as
ρ_1: the campaign by big rallies and superstar.
ρ_2: co-operating with other small political parties to reduce secured votes of the opposition.

Simultaneously W takes two strategies:
τ_1: Making lot of promises to the people.
τ_2: campaigning by use of mixed media such as publication and television.

Let us consider matrix game D with payoffs of TrIFN, where the payoff matrix of the political parties M is given as follows:

$$\widetilde{D} = \begin{bmatrix} \langle(155,165,175,180);0.7,0.2\rangle & \langle(130,146,150,165);0.6,0.2\rangle \\ \langle(75,85,95,100);0.6,0.3\rangle & \langle(160,170,184,190);0.8,0.1\rangle \end{bmatrix}$$

where the element $\langle(155,165,175,180);0.7,0.2\rangle$ in the matrix $\widetilde{D}$ is TrIFN represent that when M choose the strategy ρ_1 and W choose the strategy τ_1 then votes of the political parties M is between 155 and 180 the maximum confidence level and minimum non-confidence level of the head election exponent of $\widetilde{D}$ are 0.7 and 0.2, respectively. In this case, the hestiance degree is 0.1 other elements in $\widetilde{D}$ may be identically explained.

From eqs.(25) and (30) the parameterized linear programming is obtained as follows:

$$\max\left\{\xi\frac{\nu_\alpha^L+\nu_L^\lambda}{2}+(1-\xi)\frac{\nu_\alpha^L+\nu_L^\lambda+\nu_\alpha^R+\nu_R^\lambda}{4}\right\}$$

$$s.t.\begin{cases} \frac{(0.7-\alpha)155+165\alpha}{0.7}x_1+\frac{(0.6-\alpha)75+85\alpha}{0.6}x_2\geq\nu_\alpha^L \\ \frac{(0.6-\alpha)130+146\alpha}{0.6}x_1+\frac{(0.8-\alpha)160+170\alpha}{0.8}x_2\geq\nu_\alpha^L \\ \frac{(0.7-\alpha)180+175\alpha}{0.7}x_1+\frac{(0.6-\alpha)100+95\alpha}{0.6}x_2\geq\nu_\alpha^R \\ \frac{(0.6-\alpha)165+150\alpha}{0.6}x_1+\frac{(0.8-\alpha)190+184\alpha}{0.8}x_2\geq\nu_\alpha^R \\ \frac{(1-\lambda)165+(\lambda-0.2)155}{0.8}x_1+\frac{(1-\lambda)85+(\lambda-0.3)75}{0.7}x_2\geq\nu_L^\lambda \\ \frac{(1-\lambda)146+(\lambda-0.2)130}{0.8}x_1+\frac{(1-\lambda)170+(\lambda-0.1)160}{0.9}x_2\geq\nu_L^\lambda \\ \frac{(1-\lambda)175+(\lambda-0.2)180}{0.8}x_1+\frac{(1-\lambda)95+(\lambda-0.3)100}{0.7}x_2\geq\nu_R^\lambda \\ \frac{(1-\lambda)150+(\lambda-0.2)165}{0.8}x_1+\frac{(1-\lambda)184+(\lambda-0.1)190}{0.9}x_2\geq\nu_R^\lambda \\ x_1+x_2=1 \\ x_1\geq 0, x_2\geq 0 \end{cases} \tag{31}$$

and

$$\min\left\{\xi\frac{\omega_\alpha^R+\omega_R^\lambda}{2}+(1-\xi)\frac{\omega_\alpha^L+\omega_L^\lambda+\omega_\alpha^R+\omega_R^\lambda}{4}\right\}$$

$$s.t.\begin{cases}\dfrac{(0.7-\alpha)155+165\alpha}{0.7}y_1+\dfrac{(0.6-\alpha)130+146\alpha}{0.6}y_2\le\omega_\alpha^L\\ \dfrac{(0.6-\alpha)75+85\alpha}{0.6}y_1+\dfrac{(0.8-\alpha)160+170\alpha}{0.8}y_2\le\omega_\alpha^L\\ \dfrac{(0.7-\alpha)180+175\alpha}{0.7}y_1+\dfrac{(0.6-\alpha)165+150\alpha}{0.6}y_2\le\omega_\alpha^R\\ \dfrac{(0.6-\alpha)100+95\alpha}{0.6}y_1+\dfrac{(0.8-\alpha)190+184\alpha}{0.8}y_2\le\omega_\alpha^R\\ \dfrac{(1-\lambda)165+(\lambda-0.2)155}{0.8}y_1+\dfrac{(1-\lambda)146+(\lambda-0.2)130}{0.8}y_2\le\omega_L^\lambda\\ \dfrac{(1-\lambda)85+(\lambda-0.3)75}{0.7}y_1+\dfrac{(1-\lambda)170+(\lambda-0.1)160}{0.9}y_2\le\omega_L^\lambda\\ \dfrac{(1-\lambda)175+(\lambda-0.2)180}{0.8}y_1+\dfrac{(1-\lambda)150+(\lambda-0.2)165}{0.8}y_2\le\omega_R^\lambda\\ \dfrac{(1-\lambda)95+(\lambda-0.3)100}{0.7}y_1+\dfrac{(1-\lambda)184+(\lambda-0.1)190}{0.9}y_2\le\omega_R^\lambda\\ y_1+y_2=1\\ y_1\ge,y_2\ge0\end{cases}\tag{32}$$

respectively.

For the given $\xi=0.8$, $\alpha\in[0,0.6]$ and $\lambda\in[0.3,1]$. The greatest possible value of α and the smallest possible value of λ are computed as follows:

$$\min\{t_{\widetilde{D}_{ij}}|i=1,2;J=1,2\}=\min\{0.7,0.6,0.6,0.8\}=0.6$$

and

$$\max\{z_{\widetilde{D}_{ij}}|i=1,2;J=1,2\}=\max\{0.2,0.2,0.3,0.1\}=0.3$$

respectively.

Solving Eqs.(31) and (32) we computed by simplex method for linear programming. The result as follows:

Player M's gain-floor $\widetilde{\nu}^*$ stay in the ranges $\widetilde{\nu}^*_{\langle\alpha,\lambda\rangle}=[136.82,161.82]$. When $\langle\alpha,\lambda\rangle=\langle0.6,0.3\rangle,\widetilde{\nu}^*_{\langle\alpha,\lambda\rangle}=[148.18,157.75]$ is the most possible value of the gain-floor $\widetilde{\nu}^*$ of player M, where $x^{*T}(\alpha,\lambda)=(0.824,0.176)$. And Player W's loss-ceiling $\widetilde{\omega}^*$ stay in the ranges $\widetilde{\omega}^*_{\langle\alpha,\lambda\rangle} = [139.76, 168.57]$. When $\langle\alpha,\lambda\rangle = \langle 0.6, 0.3\rangle$, $\widetilde{\omega}^*_{\langle\alpha,\lambda\rangle} = [150.05, 159.13]$ is the most possible value of the loss-ceiling $\widetilde{\omega}^*$ of player W, where $y^{*T}(\alpha,\lambda) = (0.306, 0.694)$. Thus, the approximate values of player M's gain-floor $\widetilde{\nu}^*$ and player W's loss-ceiling $\widetilde{\omega}^*$ which are TrIFNs are $\widetilde{\nu}^* = \langle(136.82, 148.18, 157.75, 161.82); 0.6, 0.3\rangle$ and $\widetilde{\omega}^* = \langle(139.76, 150.05, 159.13, 168.57); 0.6, 0.3\rangle$, respectively.

Example 2.

From example 1 we will reduced matrix game $\widetilde{D}$ with payoffs of TrIFN to matrix game $\widetilde{D}$ of TIFN as follows:

$$\widetilde{D} = \begin{bmatrix} \langle(155,170,180);0.7,0.2\rangle & \langle(130,148,165);0.6,0.2\rangle \\ \langle(75,90,100);0.6,0.3\rangle & \langle(160,177,190);0.8,0.1\rangle \end{bmatrix}$$

From eqs.(25) and (30) the parameterized linear programming is obtained as follows:

$$\max\left\{\xi\frac{\nu^L_\alpha+\nu^\lambda_L}{2}+(1-\xi)\frac{\nu^L_\alpha+\nu^\lambda_L+\nu^R_\alpha+\nu^\lambda_R}{4}\right\}$$

$$s.t.\begin{cases} \frac{(0.7-\alpha)155+170\alpha}{0.7}x_1+\frac{(0.6-\alpha)75+90\alpha}{0.6}x_2 \ge \nu^L_\alpha \\ \frac{(0.6-\alpha)130+148\alpha}{0.6}x_1+\frac{(0.8-\alpha)160+177\alpha}{0.8}x_2 \ge \nu^L_\alpha \\ \frac{(0.7-\alpha)180+170\alpha}{0.7}x_1+\frac{(0.6-\alpha)100+90\alpha}{0.6}x_2 \ge \nu^R_\alpha \\ \frac{(0.6-\alpha)165+148\alpha}{0.6}x_1+\frac{(0.8-\alpha)190+177\alpha}{0.8}x_2 \ge \nu^R_\alpha \\ \frac{(1-\lambda)170+(\lambda-0.2)155}{0.8}x_1+\frac{(1-\lambda)90+(\lambda-0.3)75}{0.7}x_2 \ge \nu^\lambda_L \\ \frac{(1-\lambda)148+(\lambda-0.2)130}{0.8}x_1+\frac{(1-\lambda)177+(\lambda-0.1)160}{0.9}x_2 \ge \nu^\lambda_L \\ \frac{(1-\lambda)170+(\lambda-0.2)180}{0.8}x_1+\frac{(1-\lambda)90+(\lambda-0.3)100}{0.7}x_2 \ge \nu^\lambda_R \\ \frac{(1-\lambda)148+(\lambda-0.2)165}{0.8}x_1+\frac{(1-\lambda)177+(\lambda-0.1)190}{0.9}x_2 \ge \nu^\lambda_R \\ x_1+x_2=1 \\ x_1\ge 0, x_2\ge 0 \end{cases} \tag{33}$$

and

$$\min\left\{\xi\frac{\omega_\alpha^R+\omega_R^\lambda}{2}+(1-\xi)\frac{\omega_\alpha^L+\omega_L^\lambda+\omega_\alpha^R+\omega_R^\lambda}{4}\right\}$$

$$s.t.\begin{cases}\frac{(0.7-\alpha)155+170\alpha}{0.7}y_1+\frac{(0.6-\alpha)130+148\alpha}{0.6}y_2\leq\omega_\alpha^L\\ \frac{(0.6-\alpha)75+90\alpha}{0.6}y_1+\frac{(0.8-\alpha)160+177\alpha}{0.8}y_2\leq\omega_\alpha^L\\ \frac{(0.7-\alpha)180+170\alpha}{0.7}y_1+\frac{(0.6-\alpha)165+148\alpha}{0.6}y_2\leq\omega_\alpha^R\\ \frac{(0.6-\alpha)100+90\alpha}{0.6}y_1+\frac{(0.8-\alpha)190+177\alpha}{0.8}y_2\leq\omega_\alpha^R\\ \frac{(1-\lambda)170+(\lambda-0.2)155}{0.8}y_1+\frac{(1-\lambda)148+(\lambda-0.2)130}{0.8}y_2\leq\omega_L^\lambda\\ \frac{(1-\lambda)90+(\lambda-0.3)75}{0.7}y_1+\frac{(1-\lambda)177+(\lambda-0.1)160}{0.9}y_2\leq\omega_L^\lambda\\ \frac{(1-\lambda)170+(\lambda-0.2)180}{0.8}y_1+\frac{(1-\lambda)148+(\lambda-0.2)165}{0.8}y_2\leq\omega_R^\lambda\\ \frac{(1-\lambda)90+(\lambda-0.3)100}{0.7}y_1+\frac{(1-\lambda)177+(\lambda-0.1)190}{0.9}y_2\leq\omega_R^\lambda\\ y_1+y_2=1\\ y_1\geq,y_2\geq 0\end{cases}\tag{34}$$

respectively.

Solving Eqs.(33) and (34) we computed by simplex method for linear programming. The result as follows: Player M's gain-floor $\widetilde{\nu}^*$ stay in the ranges $\widetilde{\nu}^*_{\langle\alpha,\lambda\rangle}=[136.82,161.82]$. When $\langle\alpha,\lambda\rangle=\langle 0.6,0.3\rangle,\widetilde{\nu}^*_{\langle\alpha,\lambda\rangle}=152.22$ is the most possible value of the gain-floor $\widetilde{\nu}^*$ of player M, where $x^{*T}(\alpha,\lambda)=(0.806,0.194)$. And Player W's loss-ceiling $\widetilde{\omega}^*$ stay in the ranges $\widetilde{\omega}^*_{\langle\alpha,\lambda\rangle}=[139.76,168.57]$. When $\langle\alpha,\lambda\rangle=\langle 0.6,0.3\rangle$, $\widetilde{\omega}^*_{\langle\alpha,\lambda\rangle}=155.13$ is the most possible value of the loss-ceiling $\widetilde{\omega}^*$ of player W, where $y^{*T}(\alpha,\lambda)=(0.284,0.716)$. Thus, the approximate values of player M's gain-floor $\widetilde{\nu}^*$ and player W's loss-ceiling $\widetilde{\omega}^*$ which are TrIFNs are $\widetilde{\nu}^*=\langle(136.82,152.22,161.82);0.6,0.3\rangle$ and $\widetilde{\omega}^*=\langle(139.76,155.13,168.57);0.6,0.3\rangle$, respectively.

From examples 1 and 2 we will find the estimate values of player M's gain-floor $\widetilde{\nu}^*$ and player W's loss-ceiling $\widetilde{\omega}^*$ for matrix game with payoffs of TIFN and matrix game with payoffs of TrIFN are a little different. The matrix game with payoffs of TIFN is the most possible single value, while the matrix game with payoffs of TrIFN is the range the most possible.

5 Conclusion

Game theory is about the strategy of the decision of player. Good decisions require accurate and precise data. But in some cases the information available to a fuzzy uncertainty will affect the decision and payoffs.

In this work we have examples of two-person zero-sum games which only two players who have defined the strategy of each player on two strategies. To find the best response we have used the concept of the cut sets and concept of solution of matrix games with payoffs of TrIFNs. Intuitionistic fuzzy linear programming models are established for two playes, which change into bi-objective parameterized linear programming model. Two linear programming models are constructed to generate the maximin and minimax strategies for players it is seen the solving TrIFN matrix games be comes to solving a pair of intuitionistic fuzzy linear programming problems. This is one example with respect to the election. We also can apply to the issue of market share, inventory management[15],[16], finance[17], management and economics.

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