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Redox reactions as experimental examples of ternary weak algebraic hyperstructures

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Abstract

A ternary hyperoperation on a set H is a 3-ary hyperoperation, which associates a subset of H with any three elements of H. In this paper, we give examples of ternary hyperoperations associated with redox reactions. We observe that for Ag, Cu, Am and Au the ternary hyperoperations are weak associative and so their algebraic structures are H_v-semigroups.

Keywords: Algebraic hyperstructure, ternary hyperoperation, weak associativity, H_v-semigroup, redox reactions.

1 Main definitions and results related to ternary hyperoperations

Davvaz and Vougiouklis introduced the notion of a n-ary hypergroup [1], which generalizes both hypergroups and n-groups. This concept has been analyzed in other papers, by Ghadiri and Waphare [2], Leoreanu- Fotea and Davvaz [3–6], Davvaz et al. [7, 8]. In what follows, we present some basic definitions and results for $n = 3$.

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A *ternary hyperoperation* on a nonempty set H is a map $f : H \times H \times H \to \mathcal{P}^*(H)$, where $\mathcal{P}^*(H)$ denotes the set of all non-empty subsets of H. The pair (H, f) is called a *ternary hypergroupoid*. For non-empty subsets A, B, C of H, set

$$f(A, B, C) = \bigcup_{a \in A; b \in B; c \in C} f(a, b, c),$$

and for $h \in H$ write $f(h, B, C)$ instead of $f(\{h\}, B, C)$. Similarly, for $f(A, h, C)$ and $f(A, B, h)$. A ternary hypergroupoid (H, f) is a *ternary semihypergroup* if for all a_1, a_2, a_3, a_4, a_5 of H we have

$$f(f(a_1, a_2, a_3), a_4, a_5) = f(a_1, f(a_2, a_3, a_4), a_5) = f(a_1, a_2, f(, a_3, a_4, a_5)). \tag{1}$$

Moreover, if in a ternary semihypergroup (H, f) the following condition holds

$$\forall a, b, c \in H, \exists x, y, z \in H, \quad c \in f(x, a, b) \cap f(a, y, b) \cap f(a, b, z) \tag{2}$$

then (H, f) is called a *ternary hypergroup*. Ternary hyperstructures has been analyzed by Davvaz and Leoreanu-Fotea [3], Hila et al. [9, 10], Mirvakili et al [11], Yaqoob et al. [12] etc. Let us present now some examples:

EXAMPLE 1. Let $(L, \vee, \wedge)$ be a distributive lattice. For all $a_1, a_2, a_3 \in L$, define $f(a_1, a_2, a_3) = \{x \in L \mid a_1 \wedge a_2 \wedge a_3 \leq x \leq a_1 \vee a_2 \vee a_3\}$. Then, (L, f) is a ternary hypergroup.

EXAMPLE 2. Let R be an equivalence relation on a non-empty set H. For all $a_1, a_2, a_3 \in L$, define $f_R(a_1, a_2, a_3) = \{x \in H \mid (a, x) \in R$ or $(b, x) \in R$ or $(c, x) \in R\}$. Then, (H, f_R) is a ternary hypergroup.

A *ternary polygroup* is a ternary hypergroup $(P, f, e, {}^{-1})$, where f is a ternary hyperoperation on P, such that the condition (1) holds for every $a_1, ..., a_5 \in P$, $e \in P$ is a unique element such that $f(x, e, e) = f(e, x, e) = f(e, e, x) = x$ for every $x \in P$ and ${}^{-1} : P \to P$ is a unitary operation such that $e^{-1} = e$ and $z \in f(x_1, x_2, x_3)$ implies that $x_1 \in f(z, x_2^{-1}, x_3^{-1})$, $x_2 \in f(x_1^{-1}, z, x_3^{-1})$ and $x_3 \in f(x_1^{-1}, x_2^{-1}, z)$.

EXAMPLE 3. Let $P = \{e, a, b\}$ and define a commutative ternary hyperoperation f on P, as follows:

$f(a, a, b) = f(a, b, b) = P$;
$f(e, a, b) = f(a, a, a) = f(b, b, b) = \{a, b\}$;
$f(e, e, a) = a$; $f(e, e, b) = b$; $f(e, e, e) = e$;

$f(e,b,b)=\{e,a\};\ f(e,a,a)=\{e,b\}.$
Then, $(P,\cdot,e,^{-1})$ is a ternary polygroup, in which $a^{-1}=a,\ b^{-1}=b.$

If instead of equalities in (1) we have only non-empty intersections, which means that for all a_1,a_2,a_3,a_4,a_5 of H we have
$f(f(a_1,a_2,a_3),a_4,a_5)\cap f(a_1,f(a_2,a_3,a_4),a_5)\neq\emptyset;$
$f(f(a_1,a_2,a_3),a_4,a_5)\cap f(a_1,a_2,f(,a_3,a_4,a_5))\neq\emptyset;$
$f(a_1,f(a_2,a_3,a_4),a_5)\cap f(a_1,a_2,f(,a_3,a_4,a_5))\neq\emptyset,$
then (H,f) is called a *ternary H_v- semigroup.* We call the above conditions *weak associativity.* Moreover, if the condition (2) holds in a ternary H_v-semigroup H, then it is called a *ternary H_v- group.* The quotient of a group with respect to an equivalence relation is an H_v-group. A hyperoperation $(\cdot)$ on H is called *smaller* than a hyperoperation $(*)$ on the same set if there exists $f\in Aut(H,*)$ such that $x\cdot y\subseteq f(x*y)$ for all $x,y\in H$. T. Vougiouklis pointed out that a greater hyperoperation than the one of a given H_v-group defines an H_v-group, too. A hyperoperation is called *minimal* if it does not contain any other hyperoperation defined on the same set. S-C Chung and B-M Choi [13] and then R. Bayon and N. Lygeros [14] obtained that there are 13 minimal H_v-groups with a scalar unit, at order 3. Notice that if a certain hyperoperation is weak associative, then any greater hyperoperation on the same set is weak associative, too. H_v-structures were introduced by T. Vougiouklis [15] as a generalization of the well-known algebraic hyperstructures. Some axioms of classical algebraic hyperstructures are replaced by their corresponding weak axioms in H_v-structures, see [15–22]. Most of H_v-structures are used in representation theory. In [14, 23], Bayon and Lygeros counted the number of H_v-groups of certain finite sets. For instance, up to an isomorphism, there are 20 H_v–groups on a set with two elements and there are 1.026.462 H_v-groups on a set with three elements. There are 8.028.299.905 abelian H_v-groups on a set with 4 elements. Let us give now some examples of ternary H_v-groups.

Example 4. On the set R^n, where R is the set of real numbers, for all x,y,z in R^n, define the following ternary hyperoperation: $f(x,y,z)=\{r(x+y+z)\mid r\in[0,1]\}$. Then (R^n,f) is a ternary H_v- group.

Example 5. Let $(G,\cdot)$ be a group and $f:G\to G$ be a map. We define a hyperoperation (∂) on G as follows:

$$\forall x,y,z\in G,\ \partial(x,y,z)=\{f(x)yz,xf(y)z,xyf(z)\}.$$

Then, (G,∂) is a ternary H_v- group.

If $(G, \cdot)$ is a semigroup, then (G, ∂) is a ternary H_v- semigroup.

We refer the readers to [16, 24–30] to see some applications of algebraic hyperstructures.

2 Ternary redox reactions

One of motivations for the study of hyperstructures comes from chemical reactions. In [31], Davvaz and Dehghan-Nezhad provided examples of hyperstructures associated with chain reactions. In [32], Davvaz et al. introduced examples of weak hyperstructures associated with dismutation reactions. In [33], Davvaz et al. investigated the examples of hyperstructures and weak hyperstructures associated with redox reactions. Also, see [34–36].

We recall the following paragraph from [33]. Consider the redox reaction of Ag^{2+} with Ag:

$$Ag^{2+} + Ag \quad \longrightarrow \quad Ag^{+}.$$

We can write two half-reactions for this reaction:

(1) $Ag^{2+} + e \quad \longrightarrow \quad Ag^{+}$,

(2) $Ag \quad \longrightarrow \quad Ag^{+} + e$.

The E^0 of the first reaction ($E^0_{cathode}$) is 1.98 V (vs. SHE) and the E^0 of the second reaction (E^0_{anode}) is 0.799 V (vs. SHE) [37]. Therefore, in this case, the E^0_{cell} ($E^0_{cathode} - E^0_{anode} = 1.181$) is positive and the above redox reaction between Ag^{2+} and Ag is spontaneous.

Silver (Ag) is a transition metal and has a large number of applications in jewelry, electrical contacts and conductors, catalysis of chemical reactions, disinfectants and microbiocides. Silver plays a known natural biological role in humans and itself is not toxic, but most silver salts are toxic, and some may be carcinogenic. Ag can be in three oxidation state: Ag (0), Ag (I) and Ag (II). Among Ag (I) and Ag (II), Ag (I) is very well characterized and many simple ionic compounds are known containing Ag^{+}. However, AgF_2 is known which Ag has oxidation state of II in it. AgF_2 is strongly oxidizing and a good fluorimating agent. But Ag (II) is more stable in complex forms. A number of Ag (II) complexes have been obtained by oxidation of Ag (I) salts is aqueous solution in the presences of the ligand. For example, $[Ag\ (pyridine)_4]^{2+}$ and $[Ag\ (bi\ pyridine)_2]^{2+}$ are quite stable. The +1 oxidation state is the best known oxidation state of silver. Ag^{+} salts are generally insoluble in water with the exception of nitrate, fluoride and perchlorate. Most stable Ag (I) complexes have a linear structure [38].

As described above, Ag species with different oxidation states can react with themselves. Consider $\mathbb{H} = \{Ag^{2+},\ Ag^{+},\ Ag\}$. All possible ternary products for spontaneous reactions are presented in the following table (Table 1):

$f(Ag^{2+},-,-)$	Ag^{2+}	Ag^{+}	Ag
Ag^{2+}	Ag^{2+}	$Ag^{+},\ Ag^{2+}$	$Ag^{+},\ Ag^{2+}$
Ag^{+}	$Ag^{+},\ Ag^{2+}$	$Ag^{+},\ Ag^{2+}$	$Ag^{+},\ Ag^{2+}$
Ag	$Ag^{+},\ Ag^{2+}$	$Ag^{+},\ Ag^{2+}$	Ag^{+}

$f(Ag^{+},-,-)$	Ag^{2+}	Ag^{+}	Ag
Ag^{2+}	$Ag^{+},\ Ag^{2+}$	$Ag^{+},\ Ag^{2+}$	Ag^{+}
Ag^{+}	$Ag^{+},\ Ag^{2+}$	Ag^{+}	$Ag,\ Ag^{+}$
Ag	Ag^{+}	$Ag,\ Ag^{+}$	$Ag,\ Ag^{+}$

$f(Ag,-,-)$	Ag^{2+}	Ag^{+}	Ag
Ag^{2+}	Ag^{+}	$Ag,\ Ag^{+}$	$Ag,\ Ag^{+}$
Ag^{+}	$Ag,\ Ag^{+}$	$Ag,\ Ag^{+}$	$Ag,\ Ag^{+}$
Ag	$Ag,\ Ag^{+}$	$Ag,\ Ag^{+}$	Ag

Tab. 1: Redox ternary combination for Ag

Theorem 4. $(\mathbb{H}, f)$ *is a ternary H_v-semigroup.*

Proof. We must show that the weak associativity for f holds. Here, we present three cases.

Case (1):

$$\begin{aligned}
& f\Big(f(Ag^{2+},\ Ag,\ Ag^{2+}),\ Ag^{+},\ Ag^{2+}\Big) \\
&= f\Big(\{Ag^{+},\ Ag^{2+}\},\ Ag^{+},\ Ag^{2+}\Big) \\
&= f(Ag^{+},\ Ag^{+},\ Ag^{2+}) \cup f(Ag^{2+},\ Ag^{+},\ Ag^{2+}) \\
&= \{Ag^{+},\ Ag^{2+}\} \cup \{Ag^{+},\ Ag^{2+}\} \\
&= \{Ag^{+},\ Ag^{2+}\},
\end{aligned}$$

$$\begin{aligned}
&f\Big(Ag^{2+},\ f(Ag,\ Ag^{2+},\ Ag^{+}),\ Ag^{2+}\Big)\\
&= f\Big(Ag^{2+},\ \{Ag,\ Ag^{2+}\},\ \ Ag^{2+}\Big)\\
&= f(Ag^{2+},\ Ag,\ Ag^{2+}) \cup f(Ag^{2+},\ Ag^{+},\ Ag^{2+})\\
&= \{Ag^{+},\ Ag^{2+}\} \cup \{Ag^{+},\ Ag^{2+}\}\\
&= \{Ag^{+},\ Ag^{2+}\},
\end{aligned}$$

$$\begin{aligned}
&f\Big(Ag^{2+},\ Ag,\ f(Ag^{2+},\ Ag^{+},\ Ag^{2+})\Big)\\
&= f\Big(Ag^{2+},\ Ag,\ \{Ag^{+},\ Ag^{2+}\}\Big)\\
&= f(Ag^{2+},\ Ag,\ Ag^{+}) \cup f(Ag^{2+},\ Ag,\ Ag^{2+})\\
&= \{Ag^{+},\ Ag^{2+}\} \cup \{Ag^{+},\ Ag^{2+}\}\\
&= \{Ag^{+},\ Ag^{2+}\}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&f\Big(f(Ag^{2+},\ Ag,\ Ag^{2+}),\ Ag^{+},\ Ag^{2+}\Big)\\
&= f\Big(Ag^{2+},\ f(Ag,\ Ag^{2+},\ Ag^{+}),\ Ag^{2+}\Big)\\
&= f\Big(Ag^{2+},\ Ag,\ f(Ag^{2+},\ Ag^{+},\ Ag^{2+})\Big).
\end{aligned}$$

Case (2):

$$\begin{aligned}
&f\Big(f(Ag,\ Ag^{2+},\ Ag^{2+}),\ Ag,\ Ag^{+}\Big)\\
&= f\Big(Ag^{+},\ Ag,\ Ag^{+}\Big)\\
&= \{Ag,\ Ag^{+}\},
\end{aligned}$$

$$\begin{aligned}
&f\Big(Ag,\ f(Ag^{2+},\ Ag^{2+},\ Ag),\ Ag^{+}\Big)\\
&= f\Big(Ag,\ \{Ag^{+},\ Ag^{2+}\},\ Ag^{+}\Big)\\
&= f(Ag,\ Ag^{+},\ Ag^{+}) \cup f(Ag,\ Ag^{2+},\ Ag^{+})\\
&= \{Ag,\ Ag^{+}\} \cup \{Ag^{+}\}\\
&= \{Ag,\ Ag^{+}\}
\end{aligned}$$

$$\begin{aligned}
&f\Big(Ag,\ Ag^{2+},\ f(Ag^{2+},\ Ag,\ Ag^{+})\Big)\\
&= f\Big(Ag,\ Ag^{2+},\ \{Ag^{+},\ Ag^{2+}\}\Big)\\
&= f(Ag,\ Ag^{2+},\ Ag^{+}) \cup f(Ag,\ Ag^{2+},\ Ag^{2+})\\
&= \{Ag,\ Ag^{+}\} \cup \{Ag^{+}\}\\
&= \{Ag,\ Ag^{+}\}
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&f\Big(f(Ag,\ Ag^{2+},\ Ag^{2+}),\ Ag,\ Ag^{+}\Big)\\
&= f\Big(Ag,\ f(Ag^{2+},\ Ag^{2+},\ Ag),\ Ag^{+}\Big)\\
&= f\Big(Ag,\ Ag^{2+},\ f(Ag^{2+},\ Ag,\ Ag^{+})\Big).
\end{aligned}$$

Case (3): Although in Cases 1 and 2 we obtain associativity, the following case shows that only the weak associativity condition holds

$$\begin{aligned}
&f\Big(f(Ag^{2+},\ Ag^{+},\ Ag),\ Ag,\ Ag\Big)\\
&= f\Big(\{Ag^{+},\ Ag^{2+}\},\ Ag,\ Ag\Big)\\
&= \{Ag,\ Ag^{+}\} \cup \{Ag^{+}\}\\
&= \{Ag,\ Ag^{+}\},
\end{aligned}$$

$$\begin{aligned}
&f\Big(Ag^{2+},\ f(Ag^{+},\ Ag,\ Ag),\ Ag\Big)\\
&= f\Big(Ag^{2+}, \{Ag,\ Ag^{+}\},\ Ag\Big)\\
&= f(Ag^{2+}, Ag,\ Ag) \cup f(Ag^{2+},\ Ag^{+},\ Ag)\\
&= \{Ag^{+}\} \cup \{Ag^{+},\ Ag^{2+}\}\\
&= \{Ag^{+},\ Ag^{2+}\},
\end{aligned}$$

$$\begin{aligned}
&f\Big(Ag^{2+},\ Ag^{+},\ f(Ag,\ Ag,\ Ag)\Big)\\
&= f(Ag^{2+},\ Ag^{+},\ Ag)\\
&= f(Ag^{2+}, Ag,\ Ag) \cup f(Ag^{2+},\ Ag^{+},\ Ag)\\
&= \{Ag^{+},\ Ag^{2+}\}.
\end{aligned}$$

Therefore, we obtain weak associativity in this case, i.e.,

$$\begin{aligned}
&f\Big(f(Ag^{2+},\ Ag^{+},\ Ag),\ Ag,\ Ag\Big)\\
&\cap\ f\Big(Ag^{2+},\ f(Ag^{+},\ Ag,\ Ag),\ Ag\Big) \neq \emptyset,\\
&f\Big(Ag^{2+},\ f(Ag^{+},\ Ag,\ Ag),\ Ag\Big)\\
&\cap\ f\Big(Ag^{2+},\ Ag^{+},\ f(Ag,\ Ag,\ Ag)\Big) \neq \emptyset.
\end{aligned}$$

□

Copper is a chemical element with symbol (Cu) and atomic number 29. It is a ductile metal with very high thermal and electrical conductivity. Pure copper is soft and malleable; a freshly exposed surface has a reddish-orange color. It is used as a conductor of heat and electricity, a building material, and a constituent of various metal alloys.

Consider $\mathbb{K} = \{Cu,\ Cu^{+},\ Cu^{2+},\ Cu^{3+}\}$. All possible ternary products in reactions between oxidation states of Cu are indicated in the following table (Table 2).

$g(Cu,-,-)$	Cu	Cu^{+}	Cu^{2+}	Cu^{3+}
Cu	Cu	$Cu,\ Cu^{+}$	$Cu,\ Cu^{2+}$	$Cu,\ Cu^{+},\ Cu^{2+}$
Cu^{+}	$Cu,\ Cu^{+}$	$Cu,\ Cu^{+}$	$Cu,\ Cu^{+},\ Cu^{2+}$	$Cu,\ Cu^{2+}$
Cu^{2+}	$Cu,\ Cu^{2+}$	$Cu,\ Cu^{+},\ Cu^{2+}$	$Cu,\ Cu^{2+}$	$Cu,\ Cu^{+},\ Cu^{2+}$
Cu^{3+}	$Cu,\ Cu^{+},\ Cu^{2+}$	$Cu,\ Cu^{2+}$	$Cu,\ Cu^{+},\ Cu^{2+}$	$Cu^{+},\ Cu^{2+}$

$g(Cu^{+},-,-)$	Cu	Cu^{+}	Cu^{2+}	Cu^{3+}
Cu	$Cu,\ Cu^{+}$	$Cu,\ Cu^{+}$	$Cu,\ Cu^{+},\ Cu^{2+}$	$Cu^{+},\ Cu^{2+}$
Cu^{+}	$Cu,\ Cu^{+}$	Cu^{+}	$Cu^{+},\ Cu^{2+}$	$Cu^{+},\ Cu^{2+}$
Cu^{2+}	$Cu,\ Cu^{+},\ Cu^{2+}$	$Cu^{+},\ Cu^{2+}$	$Cu^{+},\ Cu^{2+}$	$Cu^{+},\ Cu^{2+}$
Cu^{3+}	$Cu^{+},\ Cu^{2+}$	$Cu^{+},\ Cu^{2+}$	$Cu^{+},\ Cu^{2+}$	Cu^{2+}

$g(Cu^{2+},-,-)$	Cu	Cu^{+}	Cu^{2+}	Cu^{3+}
Cu	$Cu,\ Cu^{2+}$	$Cu,\ Cu^{+},\ Cu^{2+}$	$Cu,\ Cu^{2+}$	$Cu^{+},\ Cu^{2+}$
Cu^{+}	$Cu,\ Cu^{+},\ Cu^{2+}$	$Cu^{+},\ Cu^{2+}$	$Cu^{+},\ Cu^{2+}$	Cu^{2+}
Cu^{2+}	$Cu,\ Cu^{2+}$	$Cu^{+},\ Cu^{2+}$	Cu^{2+}	$Cu^{2+},\ Cu^{3+}$
Cu^{3+}	$Cu^{+},\ Cu^{2+}$	Cu^{2+}	$Cu^{2+},\ Cu^{3+}$	$Cu^{2+},\ Cu^{3+}$

$g(Cu^{3+},-,-)$	Cu	Cu^{+}	Cu^{2+}	Cu^{3+}
Cu	$Cu^{+},\ Cu^{2+}$	$Cu^{+},\ Cu^{2+}$	$Cu^{+},\ Cu^{2+},\ Cu^{3+}$	$Cu^{2+},\ Cu^{3+}$
Cu^{+}	$Cu^{+},\ Cu^{2+}$	Cu^{2+}	$Cu^{2+},\ Cu^{3+}$	$Cu^{2+},\ Cu^{3+}$
Cu^{2+}	$Cu^{+},\ Cu^{2+},\ Cu^{3+}$	$Cu^{2+},\ Cu^{3+}$	$Cu^{2+},\ Cu^{3+}$	$Cu^{2+},\ Cu^{3+}$
Cu^{3+}	$Cu^{2+},\ Cu^{3+}$	$Cu^{2+},\ Cu^{3+}$	$Cu^{2+},\ Cu^{3+}$	Cu^{3+}

Tab. 2: Redox ternary combination for Cu

Theorem 5. $(\mathbb{K}, g)$ *is a ternary* H_v*-semigroup.*

Proof. We must check that the weak associativity for g holds. Here, we present two cases.

Case (1):

$$\begin{aligned}
&g\Big(g(Cu^{+},\ Cu,\ Cu^{3+}),\ Cu^{3+},\ Cu^{+}\Big)\\
&= g\Big(\{Cu^{+},\ Cu^{2+}\},\ Cu^{3+},\ Cu^{+}\Big)\\
&= g(Cu^{+},\ Cu^{3+},\ Cu^{+}) \cup g(\ Cu^{2+},\ Cu^{3+},\ Cu^{+})\\
&= \{Cu^{+},\ Cu^{2+}\} \cup \{Cu^{2+}\}\\
&= \{Cu^{+},\ Cu^{2+}\},
\end{aligned}$$

$$\begin{aligned}
&g\Big(Cu^{+},\ g(Cu,\ Cu^{3+},\ Cu^{3+}),\ Cu^{+}\Big)\\
&= g\Big(Cu^{+},\ \{Cu^{+},\ Cu^{2+}\},\ Cu^{+}\Big)\\
&= g(Cu^{+},\ Cu^{+},\ Cu^{+}) \cup g(Cu^{+},\ Cu^{2+},\ Cu^{+})\\
&= \{Cu^{+}\} \cup \{Cu^{+},\ Cu^{2+}\}\\
&= \{Cu^{+},\ Cu^{2+}\},
\end{aligned}$$

$$\begin{aligned}
&g\Big(Cu^{+},\ Cu,\ g(Cu^{3+},\ Cu^{3+},\ Cu^{+})\Big)\\
&= g\Big(Cu^{+},\ Cu,\ \{Cu^{2+},\ Cu^{3+}\}\Big)\\
&= g(Cu^{+},\ Cu,\ Cu^{2+}) \cup g(Cu^{+},\ Cu,\ Cu^{3+})\\
&= \{Cu,\ Cu^{+},\ Cu^{2+}\} \cup \{Cu^{+},\ Cu^{2+}\}\\
&= \{Cu,\ Cu^{+},\ Cu^{2+}\}.
\end{aligned}$$

Therefore, we conclude that

$$\begin{aligned}
&g\Big(g(Cu^{+},\ Cu,\ Cu^{3+}),\ Cu^{3+},\ Cu^{+}\Big)\\
&\cap\ g\Big(Cu^{+},\ g(Cu,\ Cu^{3+},\ Cu^{3+}),\ Cu^{+}\Big)\\
&\cap\ g\Big(Cu^{+},\ Cu,\ g(Cu^{3+},\ Cu^{3+},\ Cu^{+})\Big) \neq \emptyset.
\end{aligned}$$

Case (2):

$$\begin{aligned}
&g\Big(g(Cu^{2+},\ Cu^{3+},\ Cu^{+}),\ Cu^{3+},\ Cu^{+}\Big)\\
&= g\Big(\{Cu^{2+}\},\ Cu^{3+},\ Cu^{+}\Big)\\
&= \{Cu^{2+}\},
\end{aligned}$$

$$\begin{aligned}
&g\Big(Cu^{2+},\ g(Cu^{3+},\ Cu^{+},\ Cu^{3+}),\ Cu^{+}\Big)\\
&= g\Big(Cu^{2+},\ \{Cu^{2+},\ Cu^{3+}\},\ Cu^{+}\Big)\\
&= g(Cu^{2+},\ Cu^{2+},\ Cu^{+}) \cup g(Cu^{2+},\ Cu^{3+},\ Cu^{+})\\
&= \{Cu^{+},\ Cu^{2+}\} \cup \{Cu^{2+}\}\\
&= \{Cu^{+},\ Cu^{2+}\},
\end{aligned}$$

$$
\begin{aligned}
& g\Big(Cu^{2+},\ Cu^{3+},\ g(Cu^{+},\ Cu^{3+},\ Cu^{+})\Big) \\
& = g\Big(Cu^{2+},\ Cu^{3+},\ \{Cu^{+},\ Cu^{2+}\}\Big) \\
& = g(Cu^{2+},\ Cu,\ Cu^{2+}) \cup g(Cu^{2+},\ Cu^{3+},\ Cu^{2+}) \\
& = \{Cu^{+},\ Cu^{2+}\} \cup \{Cu^{2+},\ Cu^{3+}\} \\
& = \{Cu^{+},\ Cu^{2+},\ Cu^{3+}\}.
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
& g\Big(g(Cu^{2+},\ Cu^{3+},\ Cu^{+}),\ Cu^{3+},\ Cu^{+}\Big) \\
& \cap\ g\Big(Cu^{2+},\ g(Cu^{3+},\ Cu^{+},\ Cu^{3+}),\ Cu^{+}\Big) \\
& \cap\ g\Big(Cu^{2+},\ Cu^{3+},\ g(Cu^{+},\ Cu^{3+},\ Cu^{+})\Big) \neq \emptyset.
\end{aligned}
$$

□

Now, consider $\mathbb{S} = \{Am,\ Am^{2+},\ Am^{3+},\ Am^{4+}\}$. All possible ternary products for spontaneous reactions are presented in the following table (Table 3):

Theorem 6. *$(\mathbb{S}, h)$ is a ternary H_v-semigroup.*

Proof. We must show that the weak associativity for h holds. Here, we present two cases.

Case (1):

$$
\begin{aligned}
& h\Big(h(Am,\ Am^{2+},\ Am^{4+}),\ Am^{2+},\ Am^{3+}\Big) \\
& = h\Big(\{Am,\ Am^{3+}\}\ Am^{2+},\ Am^{3+}\Big) \\
& = h(Am,\ \ Am^{2+},\ Am^{3+}) \cup h(Am^{3+},\ Am^{2+},\ Am^{3+}) \\
& = \{Am,\ \ Am^{2+},\ Am^{3+}\} \cup \{Am^{2+},\ Am^{3+}\} \\
& = \{Am,\ \ Am^{2+},\ Am^{3+}\},
\end{aligned}
$$

$$
\begin{aligned}
& h\Big(Am,\ h(Am^{2+},\ Am^{4+}),\ Am^{2+}),\ Am^{3+}\Big) \\
& = h\Big(Am,\ \{Am^{2+},\ Am^{3+}\},\ Am^{3+}\Big) \\
& = h(Am,\ \ Am^{2+},\ Am^{3+}) \cup h(Am,\ Am^{3+},\ Am^{3+}) \\
& = \{Am,\ \ Am^{2+},\ Am^{3+}\} \cup \{Am,\ Am^{3+}\} \\
& = \{Am,\ \ Am^{2+},\ Am^{3+}\},
\end{aligned}
$$

$h(Am,-,-)$	Am	Am^{2+}	Am^{3+}	Am^{4+}
Am	Am	$Am,\ Am^{2+}$	$Am,\ Am^{3+}$	$Am,\ Am^{4+}$
Am^{2+}	$Am,\ Am^{2+}$	$Am,\ Am^{2+}$	$Am,\ Am^{2+},\ Am^{3+}$	$Am,\ Am^{3+}$
Am^{3+}	$Am,\ Am^{3+}$	$Am,\ Am^{2+},\ Am^{3+}$	$Am,\ Am^{3+}$	$Am,\ Am^{2+},\ Am^{3+}$
Am^{4+}	$Am,\ Am^{4+}$	$Am,\ Am^{3+}$	$Am,\ Am^{3+},\ Am^{4+}$	$Am,\ Am^{4+}$

$h(Am^{2+},-,-)$	Am	Am^{2+}	Am^{3+}	Am^{4+}
Am	$Am,\ Am^{2+}$	$Am,\ Am^{2+}$	$Am,\ Am^{2+}\ Am^{3+}$	$Am,\ Am^{2+},\ Am^{3+}$
Am^{2+}	$Am,\ Am^{2+}$	Am^{2+}	$Am^{2+},\ Am^{3+}$	$Am^{2+},\ Am^{3+}$
Am^{3+}	$Am,\ Am^{2+}\ Am^{3+}$	$Am^{2+},\ Am^{3+}$	$Am^{2+},\ Am^{3+}$	$Am^{2+},\ Am^{3+}$
Am^{4+}	$Am,\ Am^{2+}\ Am^{3+}$	$Am^{2+},\ Am^{3+}$	$Am^{2+},\ Am^{3+}$	Am^{3+}

$h(Am^{3+},-,-)$	Am	Am^{2+}	Am^{3+}	Am^{4+}
Am	$Am,\ Am^{3+}$	$Am,\ Am^{2+},\ Am^{3+}$	$Am,\ Am^{3+}$	$Am,\ Am^{2+},\ Am^{3+}$
Am^{2+}	$Am,\ Am^{2+},\ Am^{3+}$	$Am^{2+},\ Am^{3+}$	$Am^{2+},\ Am^{3+}$	Am^{3+}
Am^{3+}	$Am,\ Am^{3+}$	$Am^{2+},\ Am^{3+}$	Am^{3+}	$Am^{3+},\ Am^{4+}$
Am^{4+}	$Am,\ Am^{3+}\ Am^{4+}$	Am^{3+}	$Am^{3+},\ Am^{4+}$	$Am^{3+},\ Am^{4+}$

$h(Am^{4+},-,-)$	Am	Am^{2+}	Am^{3+}	Am^{4+}
Am	$Am,\ Am^{4+}$	$Am,\ Am^{3+},\ Am^{4+}$	$Am,\ Am^{3+},\ Am^{4+}$	$Am,\ Am^{4+}$
Am^{2+}	$Am,\ Am^{3+},\ Am^{4+}$	Am^{3+}	$Am^{3+},\ Am^{4+}$	$Am^{3+},\ Am^{4+}$
Am^{3+}	$Am,\ Am^{3+},\ Am^{4+}$	$Am^{3+},\ Am^{4+}$	$Am^{3+},\ Am^{4+}$	$Am^{3+},\ Am^{4+}$
Am^{4+}	$Am,\ Am^{4+}$	$Am^{3+},\ Am^{4+}$	$Am^{3+},\ Am^{4+}$	Am^{4+}

Tab. 3: Redox ternary combination for Am

$$
\begin{aligned}
& h\Big(Am,\ Am^{2+},\ h(Am^{4+},\ Am^{2+},\ Am^{3+})\Big) \\
& = h\Big(Am,\ Am^{2+},\ \{Am^{3+},\ Am^{4+}\Big) \\
& = h(Am,\ \ Am^{2+},\ Am^{3+}) \cup h(Am,\ Am^{2+},\ Am^{4+}) \\
& = \{Am,\ \ Am^{2+},\ Am^{3+}\} \cup \{Am,\ Am^{3+}\} \\
& = \{Am,\ \ Am^{2+},\ Am^{3+}\},
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
& h\Big(h(Am,\ Am^{2+},\ Am^{4+}),\ Am^{2+},\ Am^{3+}\Big) \\
& = h\Big(Am,\ h(Am^{2+},\ Am^{4+}),\ Am^{2+}),\ Am^{3+}\Big) \\
& = h\Big(Am,\ Am^{2+},\ h(Am^{4+}),\ Am^{2+}),\ Am^{3+})\Big).
\end{aligned}
$$

Case (2):

$$
\begin{aligned}
&h\Big(h(Am^{2+},\ Am^{3+},\ Am^{4+}),\ Am^{4+},\ Am^{4+}\Big)\\
&= h\Big(\{Am^{2+},\ Am^{3+}\},\ Am^{4+},\ Am^{4+}\Big)\\
&= h(Am^{2+},\ \ Am^{4+},\ Am^{4+}) \cup h(Am^{3+},\ Am^{4+},\ Am^{4+})\\
&= \{Am^{3+}\} \cup \{Am^{3+},\ Am^{4+}\}\\
&= \{Am^{3+},\ Am^{4+}\},
\end{aligned}
$$

$$
\begin{aligned}
&h\Big(Am^{2+},\ h(Am^{3+},\ Am^{4+},\ Am^{4+}),\ Am^{4+}\Big)\\
&= h\Big(Am^{2+},\ \{Am^{3+}\ Am^{4+}\},\ Am^{4+}\Big)\\
&= h(Am^{2+},\ \ Am^{3+},\ Am^{4+}) \cup h(Am^{2+},\ Am^{4+},\ Am^{4+})\\
&= \{Am^{2+},\ Am^{3+}\} \cup \{Am^{3+}\}\\
&= \{Am^{2+},\ Am^{3+}\},
\end{aligned}
$$

$$
\begin{aligned}
&h\Big(Am^{2+},\ Am^{3+}, h(Am^{4+},\ Am^{4+},\ Am^{4+})\Big)\\
&= h\Big(Am^{2+},\ Am^{3+},\ \{Am^{4+}\}\Big)\\
&= h(Am^{2+},\ Am^{3+},\ Am^{4+})\\
&= \{Am^{2+},\ Am^{3+}\},
\end{aligned}
$$

Therefore, we conclude that

$$
\begin{aligned}
&h\Big(h(Am^{2+},\ Am^{3+},\ Am^{4+}),\ Am^{4+},\ Am^{4+}\Big)\\
&\cap\ h\Big(Am^{2+},\ h(Am^{3+},\ Am^{4+},\ Am^{4+}),\ Am^{4+}\Big)\\
&\cap\ h\Big(Am^{2+},\ Am^{3+}, h(Am^{4+},\ Am^{4+}),\ Am^{4+})\Big) \neq \emptyset.
\end{aligned}
$$

□

Finally, consider $\mathbb{M} = \{Au,\ Au^{+},\ Au^{2+},\ Am^{3+}\}$. All possible ternary products for spontaneous reactions are presented in the following table (Table 4):

Theorem 7. $(\mathbb{M}, \varphi)$ *is a ternary* H_v*-semigroup.*

Proof. With a similar calculation, we observe that φ is a weak associative. □

$\varphi(Au,-,-)$	Au	Au^+	Au^{2+}	Au^{3+}
Au	Au	$Au,\ Au^+$	$Au,\ Au^+$	$Au,\ Au^{3+}$
Au^+	$Au,\ Au^+$	$Au,\ Au^+$	$Au,\ Au^+$	$Au,\ Au^+,\ Au^{3+}$
Au^{2+}	$Au,\ Au^+$	$Au,\ Au^+$	Au^+	$Au,\ Au^+,\ Au^{3+}$
Au^{3+}	$Au,\ Au^{3+}$	$Au,\ Au^+,\ Au^{3+}$	$Au,\ Au^+,\ Au^{3+}$	$Au,\ Au^{3+}$

$\varphi(Au^+,-,-)$	Au	Au^+	Au^{2+}	Au^{3+}
Au	$Au,\ Au^+$	$Au,\ Au^+$	Au^+	$Au,\ Au^+,\ Au^{3+}$
Au^+	$Au,\ Au^+$	Au^+	$Au^+,\ Au^{2+}$	$Au^+,\ Au^{3+}$
Au^{2+}	Au^+	$Au^+,\ Au^{2+}$	$Au^+,\ Au^{2+}$	$Au,\ Au^{2+},\ Au^{3+}$
Au^{3+}	$Au,\ Au^+,\ Au^{3+}$	$Au^+,\ Au^{3+}$	$Au,\ Au^{2+},\ Au^{3+}$	$Au^+,\ Au^{3+}$

$\varphi(Au^{2+},-,-)$	Au	Au^+	Au^{2+}	Au^{3+}
Au	Au^+	$Au^+,\ Au^{2+}$	$Au^+,\ Au^{2+}$	$Au,\ Au^{2+},\ Au^{3+}$
Au^+	$Au^+,\ Au^{2+}$	$Au^+,\ Au^{2+}$	$Au^+,\ Au^{2+}$	$Au^+,\ Au^{2+},\ Au^{3+}$
Au^{2+}	$Au^+,\ Au^{2+}$	$Au^+,\ Au^{2+}$	$Au^+,\ Au^{2+}$	$Au^{2+},\ Au^{3+}$
Au^{3+}	$Au^+,\ Au^{2+},\ Au^{3+}$	$Au^+,\ Au^{2+},\ Au^{3+}$	$Au^{2+},\ Au^{3+}$	$Au^{2+},\ Au^{3+}$

$\varphi(Au^{2+},-,-)$	Au	Au^+	Au^{2+}	Au^{3+}
Au	$Au,\ Au^{3+}$	$Au,\ Au^+,\ Au^{3+}$	$Au^+,\ Au^{3+}$	$Au,\ Au^{3+}$
Au^+	$Au,\ Au^+,\ Au^{3+}$	$Au^+,\ Au^{3+}$	$Au^+,\ Au^{2+},\ Au^{3+}$	$Au^+,\ Au^{3+}$
Au^{2+}	$Au^+,\ Au^{3+}$	$Au^+,\ Au^{2+},\ Au^{3+}$	$Au^{2+},\ Au^{3+}$	$Au^{2+},\ Au^{3+}$
Au^{3+}	$Au,\ Au^{3+}$	$Au^+,\ Au^{3+}$	$Au^{2+},\ Au^{3+}$	Au^{3+}

Tab. 4: Redox ternary combination for Au

3 Conclusions

After the introduction of H_v-structures by Vougiouklis, there have been a number of generalizations of this fundamental concept. Ternary H_v-semigroups are algebraic structures with one associative ternary hyperoperation. We can consider ternary H_v-semigroups as a generalization of ternary semigroups and semihypergroups. In this paper, we observed that redox reactions are excellent example of ternary H_v-semigroups.

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