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Stability and square integrability of solutions of nonlinear fourth order differential equations

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Abstract

The aim of the present paper is to establish a new result, which guarantees the asymptotic stability of zero solution and square integrability of solutions and their derivatives to nonlinear differential equations of fourth order.hg

Keywords: Lyapunov functional, differential equations of fourth order, uniform asymptotic stability, square integrability.

1 Introduction

Today, in applied sciences, practical problems concerning the reaction and diffusion of chemicals, the dynamics of population in biology, the development and treatment of diseases in medicine, or the flow of fluid or gas, which has applications ranging from biology to meteorology to engineering technique fields are associated with order differential equations.

Asymptotic properties of solutions of higher differential equations are very important in the theory and applications of nonlinear differential equations and have been subject of intensive studying in the literature for example ([1]

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- [26]). Such a problems have been studied mostly for second order nonlinear differential equations by many authors. By comparison, the study of these properties for third and fourth order equations has received considerably less attention in the literature. Liapunov's method has proved to be a popular and useful technique in the study of the stability and boundedness of solutions of higher order non-linear differential equations. Excellent sources for background discussions on this technique can be found in the monographs [2], [3], and [26].

In this article, we investigate some asymptotic properties of solutions of the fourth order nonlinear differential equation

$$(g(x)\,x'')'' + a(t)\,x''' + b(t)\,x'' + c(t)\,x' + d(t)\,h(x) = 0, \tag{1}$$

where $a(t), b(t), c(t), d(t)$ and $h(x)$ are continuously differentiable functions depending only on the arguments shown, and $g(x)$ is twice continuously differentiable.

The prototype for equation (1) is a result by Cartwright [4] who established sufficient conditions to ensure, the asymptotic stability of zero solution of the following fourth order differential equation

$$x'''' + a_1 x''' + a_2 x'' + a_3 x' + f(x) = 0. \tag{2}$$

In 1989 Andres and Vlček [1], considered the same equation (2) and find sufficient conditions for the square integrability of solutions.

The problem of interest here is to investigate conditions under which all solutions of (1) converge to zero and are square integrable. The form of the equation we consider here is more general than those considered by [4] and [1]. Clearly the equation discussed in [1] and [4] is a special case of equation (1) when $g(x) = 1$ and $a(t) = a_1,\ b(t) = a_2,\ c(t) = a_3$. To the best of our knowledge the boundedness and square integrability of solutions of the above differential equation have not been discussed in the literature. Our aim here is to constitute an initial on the topic for nonautonomous differential equations of fourth order. We shall use appropriate Lyapunov function and impose suitable conditions on the functions $g(x)$, and $h(x)$.

The paper is organized as follows, in the first part of section 2, as well as several applications of these results, we establish sufficient conditions under which all solutions of the fourth-order nonlinear differential equation (1) converge to zero as $t \to \infty$ and in the second part we introduced theorem, which deal with the square integrability of solutions of the equation (1). In proving this, we shall show that the derivatives are also square integrable.

2 Assumptions and main results

We shall state here some assumptions which will be used on the functions that appeared in equation (1), and suppose that there are positive constants $a_0, b_0, c_0, d_0, g_0, a_1, b_1, c_1, d_1, g_1, h_0, m, M, \delta, \delta_0, \eta_1$ and η_2 such that the following conditions hold

i) $0 < a_0 \le a(t) \le a_1,\ 0 < b_0 \le b(t) \le b_1,\ 0 < c_0 \le c(t) \le c_1,\ 0 < d_0 \le d(t) \le d_1$ for $t \ge 0$.

ii) $0 < g_0 \le g(x) \le g_1$ for all x, $0 < m < \min\{g_0, 1\}$, and $M > \max\{g_1, 1\}$.

iii) $h(0) = 0$, $\dfrac{h(x)}{x} \ge \delta > 0$ for $x \neq 0$.

iv) $\dfrac{h_0}{m} - \dfrac{a_0\delta_0}{Md_1} \le h'(x) \le \dfrac{h_0}{2M}$ for $x \in \mathbb{R}$.

v) $b_0 > \max(\kappa_1, \kappa_2)$ where

$$\begin{cases} \kappa_1 = \dfrac{c_1 M^2}{a_0 m} + \dfrac{a_1 h_0 d_1 M}{c_0 m^3} + \dfrac{a_0 a_1 (M-1)}{M} + \dfrac{M^2 \delta_0}{a_0 m}, \\ \kappa_2 = \dfrac{2a_0 d_1 h_0}{c_0 m(M-1)} \left(\dfrac{1}{m} - \dfrac{1}{M}\right)^2 + \dfrac{2c_0 m M}{a_0} + \dfrac{2a_1 h_0 d_1}{c_0 m^3} + \\ \qquad \dfrac{c_0 c_1 (M^2+2) m}{d_1 h_0}. \end{cases}$$

The following lemma will be useful in the proof of next theorem.

Lemma 1. *(see [7]). Let* $h(0) = 0,\ xh(x) > 0\ (x \neq 0)$ *and* $\delta(t) - h'(x) \ge 0\ (\delta(t) > 0)$, *then*

$$2\delta(t)H(x) \ge h^2(x) \qquad \text{where} \quad H(x) = \int_0^x h(s)ds.$$

The first main result in this paper establishes sufficient conditions under which all solutions of the fourth order nonlinear differential equation (1) and their first, second and third derivatives converge to zero as $t \to 0$.

Theorem 2. *Further to assumptions (i)-(v), assume that there are positive constants η_1 and η_2 such that the following conditions are satisfied*

H1) $$\int_0^{+\infty} \left(|a'(t)| + |b'(t)| + |c'(t)| + |d'(t)|\right) dt \leq \eta_1.$$

H2) $$\int_{-\infty}^{+\infty} |g'(u)|\, du \leq \eta_2.$$

Then any solution x of equation (1) satisfies

$$x(t) \to 0, \quad x'(t) \to 0, \quad x''(t) \to 0, \quad x'''(t) \to 0, \quad \textit{as} \ \ t \to \infty. \tag{3}$$

Proof. We write equation (1) as the equivalent system

$$\begin{cases} x' = y, \\ y' = \dfrac{1}{g(x)} z, \\ z' = w, \\ w' = -\dfrac{a(t)}{g(x)} w + \left(R(t)\, a(t) - \dfrac{b(t)}{g(x)}\right) z - c(t)\, y - d(t)\, h(x), \end{cases} \tag{4}$$

where $R(t) = \dfrac{g'(x(t))}{g^2(x(t))} x'(t)$. Our main tool is the continuously differentiable function $W = W(t, x, y, z, w)$ defined by

$$W = e^{-\frac{1}{\eta}\int_0^t \gamma(s)\, ds} V, \tag{5}$$

where

$\gamma(t) = |a'(t)| + |b'(t)| + |c'(t)| + |d'(t)| + |R(t)|,$

and

$$\begin{aligned} 2V &= 2\beta d(t)\, H(x) + c(t)\, g(x)\, y^2 + \alpha \frac{b(t)}{g(x)} z^2 + \frac{a(t)}{g(x)} z^2 + 2\beta \frac{a(t)}{g(x)} yz \\ &+ 2\alpha d(t)\, h(x)\, z + \big[\beta b(t) - \alpha h_0 d(t)\big] y^2 - \beta \frac{1}{g(x)} z^2 + \alpha w^2 \\ &+ 2d(t)\, g(x)\, h(x)\, y + 2\alpha c(t)\, yz + 2\beta yw + 2zw, \end{aligned}$$

such that

$$H(x) = \int_0^x h(s) ds, \ \ \alpha = \frac{M}{a_0} + \epsilon \text{ and }, \ \beta = \frac{d_1 h_0}{c_0 m} + \epsilon.$$

ϵ and η are positive constants to be determined later in the proof. We rewrite $2V$ as

$$\begin{aligned} 2V &= 2\epsilon d(t)H(x) + a(t)\left[\frac{w}{a(t)} + z + \beta\frac{1}{g(x)}y\right]^2 \\ &+ c(t)\left[\frac{d(t)h(x)}{c(t)} + y + \alpha z\right]^2 \\ &+ c(t)\left[\Big(g(x)-1\Big)y + \frac{d(t)h(x)}{c(t)}\right]^2 \\ &+ V_1 + V_2 + V_3, \end{aligned}$$

where

$$\begin{aligned} V_1 &= 2d(t)\int_0^x h(s)\left[\frac{d_1 h_0}{c_0 m} - 2\frac{d(t)}{c(t)}h'(s)\right]ds, \\ V_2 &= \left[\alpha\frac{b(t)}{g(x)} - \beta\frac{1}{g(x)} - \alpha^2 c(t) + a(t)\left(\frac{1}{g(x)} - 1\right)\right]z^2, \\ V_3 &= \left[\beta b(t) - \alpha h_0 d(t) - \beta^2\frac{a(t)}{g^2(x)} - c(t)\left(g^2(x) - 3g(x) + 2\right)\right]y^2 \\ &+ \left[\alpha - \frac{1}{a(t)}\right]w^2 + 2\beta\left[1 - \frac{1}{g(x)}\right]yw. \end{aligned}$$

To prove that V is positive definite it suffices to show that V_1, V_2 and V_3 are positive. Set

$$\epsilon < \min\left\{\frac{M}{a_0}\ ,\ \frac{d_1 h_0}{c_0 m}\ ,\ \frac{m^2(b_0-\kappa_1)}{M(a_1+c_1 m)}\right\}, \tag{6}$$

then

$$\frac{M}{a_0} < \alpha < 2\frac{M}{a_0}, \qquad \frac{d_1 h_0}{c_0 m} < \beta < 2\frac{d_1 h_0}{c_0 m}. \tag{7}$$

Using conditions (i) $\sim$ (v), and inequalities (6), (7) we obtain

$$\begin{aligned} V_1 &\geq 2d(t)\int_0^x h(s)\frac{d_1}{c_0 m}\left[h_0 - 2mh'(s)\right]ds \\ &\geq 4\frac{d_0 d_1}{c_0}\int_0^x h(s)\left[\frac{h_0}{2M} - h'(s)\right]ds \geq 0. \end{aligned}$$

Rearrange V_2 we obtain the estimate

$$\begin{aligned}
V_2 &= \alpha\left[\frac{b\,(t)}{g\,(x)} - \beta\frac{a\,(t)}{g\,(x)} - \alpha c\,(t) - \frac{a\,(t)}{\alpha}\left(1 - \frac{1}{g\,(x)}\right)\right]z^2 + \beta\left[\alpha\frac{a\,(t)}{g\,(x)} - \frac{1}{g\,(x)}\right]z^2 \\
&\geq \alpha\left[\frac{b\,(t)}{g\,(x)} - \big(\frac{d_1h_0}{c_0m} + \epsilon\big)\frac{a\,(t)}{g\,(x)} - \big(\frac{M}{a_0} + \epsilon\big)c\,(t) - \frac{a_0}{M}a(t)\left(1 - \frac{1}{g\,(x)}\right)\right]z^2 \\
&+ \frac{\beta}{g(x)}\left[\frac{Ma(t)}{a_0} - 1\right]z^2 \\
&\geq \alpha\left[\frac{b_0}{M} - \frac{a_1d_1h_0}{c_0m^2} - \frac{c_1M}{a_0} - \frac{a_0a_1}{M^2}\big(M-1\big) - \epsilon\big(\frac{a_1}{m} + c_1\big)\right]z^2 \\
&\geq \frac{\alpha}{Mm}\left[m(b_0 - \kappa_1) - \epsilon M\,(a_1 + c_1m)\right]z^2 \geq 0.
\end{aligned}$$

We have also,

$$\begin{aligned}
V_3 &\geq \beta\left[b_0 - 2\frac{M}{a_0}c_0m - 2a_1\frac{d_1h_0}{c_0m^3} - \frac{c_0c_1m\,(M^2+2)}{d_1h_0}\right]y^2 \\
&+ \left[\frac{M-1}{a_0}\right]w^2 + 2\beta\left[1 - \frac{1}{g\,(x)}\right]yw \\
&\geq \psi(y,\omega),
\end{aligned}$$

where

$$\psi(y,\omega) = \frac{2\beta d_1h_0a_0}{c_0m\,(M-1)}\left[\frac{1}{M} - \frac{1}{m}\right]^2 y^2 + \left[\frac{M-1}{a_0}\right]w^2 + 2\beta\left[1 - \frac{1}{g\,(x)}\right]yw.$$

We claim that $\psi(y,\omega)$ is positive definite. To show this we calculate the discriminant

$$\triangle = \beta^2\left[1 - \frac{1}{g\,(x)}\right]^2 - \frac{2\beta d_1h_0}{c_0m}\left[\frac{1}{M} - \frac{1}{m}\right]^2.$$

From condition (ii) we have

$$\frac{1}{M} < \frac{1}{g(x)} < \frac{1}{m}, \text{ and } \frac{1}{M} < 1 < \frac{1}{m},$$

it follows that

$$\left|1 - \frac{1}{g\,(x)}\right| < \frac{1}{m} - \frac{1}{M}.$$

From (7) we see that

$$\triangle < \beta\left[\frac{2d_1h_0}{c_0m}\left(\frac{1}{M}-\frac{1}{m}\right)^2-\frac{2d_1h_0}{c_0m}\left(\frac{1}{M}-\frac{1}{m}\right)^2\right]=0.$$

Thus there exists positive number D_0 such that

$$2V \geq D_0\left(y^2+z^2+w^2+H(x)\right). \tag{8}$$

By Lemma 1 and conditions (iii) and (iv) we conclude that there exists a positive number D_1 such that

$$2V \geq D_1\left(x^2+y^2+z^2+w^2\right), \tag{9}$$

thus V is positive definite. Then we can find positive definite functions $U_1(\|X\|)$ and $U_2(\|X\|)$ such that $U_1(\|X\|) \leq V \leq U_2(\|X\|)$. By (ii) and (vii), we get

$$\int_0^t |R(s)|\,ds = \int_{\alpha_1(t)}^{\alpha_2(t)} \frac{|g'(u)|}{g^2(u)}du \leq \frac{1}{m^2}\int_{-\infty}^{+\infty}|g'(u)|\,du \leq \frac{\eta_2}{m^2} < \infty, \tag{10}$$

where $\alpha_1(t)=\min\{x(0),x(t)\}$, and $\alpha_2(t)=\max\{x(0),x(t)\}$. By (5) and inequalities, (10) and (9) we obtain

$$W \geq D_2(x^2+y^2+z^2+w^2), \tag{11}$$

where $D_2=\dfrac{D_1}{2}e^{-\frac{1}{\eta}(\eta_1+\frac{\eta_2}{m^2})}$. Therefore by conditions (H1) and (H2) we can find positive definite functions $W_1(\|X\|)$ and $W_2(\|X\|)$ such that $W_1(\|X\|) \leq W \leq W_2(\|X\|)$.

Now we prove that $\dot{W}$ is negative definite function. Along any solution $(x(t),y(t),z(t),w(t))$ of system (4), we have

$$2\dot{V}_{(4)} = -2\epsilon c(t)\,y^2+V_4+V_5+V_6+V_7+2\frac{\partial V}{\partial t},$$

where

$V_4=-2\left(\dfrac{d_1h_0}{c_0m}c(t)-d(t)\,g(x)\,h'(x)\right)y^2-2\alpha d(t)\left(\dfrac{h_0}{g(x)}-h'(x)\right)yz,$

$$V_5 = -2\left(\frac{b\,(t)}{g\,(x)} - \alpha\frac{c\,(t)}{g\,(x)} - \beta\frac{a\,(t)}{g^2\,(x)}\right)z^2,\;\; V_6 = -2\left(\alpha\frac{a\,(t)}{g\,(x)} - 1\right)w^2,$$

and

$$\begin{aligned} V_7 \;&=\; R\,(t)\Bigg(\Big(a\,(t) - \alpha b\,(t) + \beta\Big)z^2 + c\,(t)\,g^2\,(x)\,y^2 \\ &\quad +2d\,(t)\,g^2\,(x)\,h\,(x)\,y + 2\alpha a\,(t)\,zw\Bigg). \end{aligned}$$

Using again conditions (i), (ii), (iv), (v), and inequalities (6), (7) we get

$$\begin{aligned} V_4 &\le -2\left[d\,(t)\,h_0 - d\,(t)\,g\,(x)\,h'\,(x)\right]y^2 - 2\alpha d\,(t)\left[\frac{h_0}{g\,(x)} - h'\,(x)\right]yz \\ &\le -2d\,(t)\,g\,(x)\left[\frac{h_0}{g\,(x)} - h'\,(x)\right]y^2 - 2\alpha d\,(t)\left[\frac{h_0}{g\,(x)} - h'\,(x)\right]yz \\ &\le -2d\,(t)\,m\left[\frac{h_0}{g\,(x)} - h'\,(x)\right]\left[\left(y + \frac{\alpha}{2m}z\right)^2 - \left(\frac{\alpha}{2m}z\right)^2\right] \\ &\le -2d\,(t)\,m\left[\frac{h_0}{M} - h'\,(x)\right]\left(y + \frac{\alpha}{2m}z\right)^2 + 2d(t)m\left[\frac{h_0}{m} - h'\,(x)\right]\left(\frac{\alpha}{2m}z\right)^2 \\ &\le \frac{\alpha^2}{2m}d\,(t)\left[\frac{h_0}{m} - h'\,(x)\right]z^2. \end{aligned}$$

Hence,

$$\begin{aligned} V_4 + V_5 &\le -2\left[\frac{b_0}{M} - \Big(\frac{M}{a_0} + \epsilon\Big)\frac{c_1}{m} - \Big(\frac{d_1 h_0}{c_0 m} + \epsilon\Big)\frac{a_1}{m^2} - \frac{\alpha^2}{4m}\left(\frac{a_0\delta_0}{M}\right)\right]z^2 \\ &\le -2\left[\frac{b_0}{M} - \frac{M}{a_0 m}c_1 - \frac{d_1 h_0 a_1}{c_0 m^3} - \frac{M\delta_0}{a_0 m} - \frac{\epsilon}{m}\left(\frac{a_1}{m} + c_1\right)\right]z^2 \\ &\le -\frac{2}{Mm^2}\big[m^2(b_0 - \kappa_1) - \epsilon M\,(a_1 + c_1 m)\,\big]z^2 \le 0. \end{aligned}$$

We have also,

$$V_6 \le -2\left[\alpha\frac{a_0}{M} - 1\right]w^2 = -2\epsilon\frac{a_0}{M}w^2 \le 0.$$

We conclude that, there exists a positive constant λ_1 where,

$$-2\epsilon c(t)\, y^2 + V_4 + V_5 + V_6 \le -2\lambda_1 \left(y^2 + z^2 + w^2\right).$$

By (8), Cauchy Schawrtz inequality and Lemma 1 we get

$$V_7 \le |R(t)| \Bigg(\Big(a(t) + \alpha b(t) + \beta\Big) z^2 + c(t)\, g^2(x)\, y^2 + d(t)\, g^2(x) \left(h^2(x) + y^2\right) + \alpha a(t) \left(z^2 + w^2\right) \Bigg).$$

Thus

$$V_7 \le \lambda_2 |R(t)| \left(y^2 + z^2 + w^2 + H(x)\right) \le 2\frac{\lambda_2}{D_0} |R(t)|\, V,$$

where,

$$\lambda_2 = \max\left\{a_1(1+\alpha) + \alpha b_1 + \beta, (c_1 + d_1)M^2, d_1 h_0 M\right\}.$$

Next we have

$$\begin{aligned} 2\frac{\partial V}{\partial t} &= d'(t)\left[2\beta H(x) - \alpha h_0 y^2 + 2g(x)\, h(x)\, y + 2\alpha h(x)\, z\right] \\ &\quad + b'(t)\left[\alpha \frac{1}{g(x)} z^2 + \beta y^2\right] + c'(t)\left[g(x)\, y^2 + 2\alpha y z\right] \\ &\quad + a'(t)\left[\frac{1}{g(x)} z^2 + 2\beta \frac{1}{g(x)} y z\right] \\ &\le \lambda_3 \Big(|a'(t)| + |b'(t)| + |c'(t)| + |d'(t)| \Big) \left(y^2 + z^2 + w^2 + H(x)\right) \\ &\le 2\frac{\lambda_3}{D_0} \Big(|a'(t)| + |b'(t)| + |c'(t)| + |d'(t)| \Big) V, \end{aligned}$$

where

$$\lambda_3 = \max\left\{\frac{1}{m}(\alpha + \beta + 1), \frac{\beta}{m} + \alpha(1 + h_0) + M, 2\beta + h_0(1+\alpha)\right\}.$$

Taking $\dfrac{1}{\eta} = \dfrac{1}{D_0} \max\{\lambda_2,\ \lambda_3\}$, we obtain

$$\dot{V}_{(4)} \le -\lambda_1 \left(y^2 + z^2 + w^2\right) + \frac{1}{\eta}\gamma(t)V. \tag{12}$$

From conditions (H1), (H2) and inequalities (10), (12) we get

$$\begin{aligned} \dot{W}_{(4)} &= \left(\dot{V}_{(4)} - \frac{1}{\eta}\gamma(t)\,V\right) e^{-\frac{1}{\eta}\int_0^t \gamma(s)\,ds} \\ &\le -\lambda_1 \left(y^2 + z^2 + w^2\right) e^{-\frac{1}{\eta}\int_0^t \gamma(s)\,ds} \\ &\le -D_3 \left(y^2 + z^2 + w^2\right), \end{aligned} \tag{13}$$

where $D_3 = \lambda_1 e^{-\frac{1}{\eta}(\eta_1 + \frac{\eta_2}{m^2})}$. It can also be followed from [[2], Theorem 4.1.14] that the solution $(x(t), y(t), z(t), w(t))$ of system (4) is uniformly stable.
Now $E = \{(x, y, z, w) : \dot{W}_{(4)}(t, x, y, z, w) = 0\} = \{(x, 0, 0, 0) : x \in \mathbb{R}\}$ and the largest invariant set contained in E is $M = \{(0, 0, 0, 0)\}$. By LaSalle's Invariance Principle (see [8]) $(x(t), y(t), z(t), w(t)) \to (0, 0, 0, 0)$ as $t \to \infty$. This completes the proof of the Theorem 2.

Remark 1. *From Theorem 2, it follows that the solution $(x(t), x'(t), x''(t), x'''(t))$ of (1) is uniformly asymptotically stable. Then, there exists a positive constant D_3 such that*

$$|x(t)| \le D_3, \quad |x'(t)| \le D_3, \quad |x''(t)| \le D_3, \quad |x'''(t)| \le D_3, \qquad \textit{for all} \ \ t \ge 0. \tag{14}$$

Remark 2. *Tunç [17] and [18] discussed the stability of solutions of the following third order differential equations*

$$x'''' + \varphi(x'')x''' + f(x, x')x'' + G(x, x') + h(x) = 0, \tag{15}$$

$$x'''' + \varphi(x'')x''' + f(x, x', x'') + G(x, x') + h(x) = 0. \tag{16}$$

If $a(t) = b(t) = c(t) = 1$, equation (1) takes the form

$$x'''' + k_1(x, x')x''' + k_2(x, x', x'')x'' + k_3(x, x') + k_4(x) = 0, \tag{17}$$

where

$$\begin{aligned} k_1(x,x') &= \frac{1+2g'(x)}{g(x)},\ k_2(x,x',x'')=\frac{g''(x)x'^2+g'(x)x''}{g(x)}, \\ k_3(x,x') &= \frac{1}{g(x)}x',\ k_4(x)=\frac{h(x)}{g(x)}. \end{aligned}$$

1) It is clear that equations (15) and (17) are different, since the term $\varphi(x'')x'''$ appearing in equation (15) don't appear in (17) it is replaced by different term $k_1(x,x')x'''$.

2) In equation (15) the boundedness of f is needed. However in our theorem this latter condition is not required since k_2 is not Supposed bounded.

3) In equation (16) the differentiability of f on x is needed, which implies the use of the second derivative and the third derivative of g. However in our theorem this latter conditions are not required since we just need to deal with g'.

Our next result concerns the square integrability of solutions of equation (1).

Theorem 3. *In addition to conditions of Theorem 2, if we assume that, there exists a positive constant L_0 such that, $\lim_{u\to 0}|g'(u)| \le L_0$, then all the solutions of (1) and their derivatives are elements of $L^2[0,+\infty)$.*

proof. Define

$$F = F(t,x,y,z,w) = W + \rho\int_0^t \left(y^2(s)+z^2(s)+w^2(s)\right) ds,$$

where $\rho > 0$ and $W = W(t,x,y,z,w)$ is already positive definite by (5). Then F is positive definite. From (13) and choosing $\rho \le D_2$, we obtain

$$\begin{aligned} \dot{F}_{(4)} &= \dot{W}_{(4)} + \rho\left(y^2(t)+z^2(t)+w^2(t)\right) \\ &\le (-D_2+\rho)\left(y^2(t)+z^2(t)+w^2(t)\right) \le 0. \end{aligned}$$

We can conclude that, there exists a positive constant L_1 such that,

$$0 \le \lim_{t\to\infty} F(t,x(t),y(t),z(t),w(t)) \le L_1.$$

Therefore

$$\int_0^\infty y^2(s)ds \le L_1 \quad , \quad \int_0^\infty z^2(s) \le L_1 \text{ and } \int_0^\infty w^2(s)ds \le L_1. \qquad (18)$$

Clearly (18), (ii) and (4) gives

$$\int_0^\infty x'^2(s)ds \le L_1 \quad , \quad \int_0^\infty x''^2(s)ds \le \frac{1}{m^2}\int_0^\infty z^2(s)ds \le \frac{L_1}{m^2} = L_2. \qquad (19)$$

On the other hand, using the conditions of Theorem 3 and (3), it follows that

$$R^2(t) = \left[\frac{g'(x(t))}{g^2(x(t))}x'(t)\right]^2 \to 0 \quad \text{as} \quad t \to \infty.$$

Then, there exists a positive constant k such that $R^2(t) \le k$ for all $t \ge 0$. Using Cauchy Schwartz inequality and since $x'''(t) = \dfrac{1}{g(x(t))}w(t) - R(t)z(t)$, we get

$$\begin{aligned}\int_0^\infty x'''^2(s)ds &= \int_0^\infty \frac{w^2(s)}{g^2(x(s))}ds + \int_0^\infty R^2(s)z^2(s)ds \\ &\quad -2\int_0^\infty \frac{R(s)}{g(x(s))}z(s)w(s)ds \\ &\le 2\int_0^\infty \frac{w^2(s)}{g^2(x(s))}ds + 2\int_0^\infty R^2(s)z^2(s)ds \\ &\le 2N\int_0^\infty (z^2(s)+w^2(s))ds \le 4NL_1 = L_3,\end{aligned}$$

where $N = \max\{\frac{1}{m^2}, k^2\}$. Multiplying (1) by $x(t)$ and integrating by parts, from 0 to t all the terms on the LHS, we obtain

$$\int_0^t d(s)x(s)h(x(s))ds = I_1(t) + I_2(t) + I_3(t) + I_4(t) + L, \qquad (20)$$

where

$$I_1(t) = -\Big(g(x(t))x''(t)\Big)''x(t) + g(x(t))x''(t)x'(t) - \int_0^t g(x(s))x''^2(s)ds,$$

$$I_2(t) = -a(t)x''(t)x(t) + \frac{a(t)}{2}x'^2(t) + \int_0^t a'(s)\Big(x(s)x''(s) - \frac{1}{2}x'^2(s)\Big)ds,$$

$$I_3(t) = -b(t)\big(x(t)x'(t) + \int_0^t x'^2(s)ds\big) + \int_0^t b'(s)\Big(x(s)x'(s) - \int_0^s x'^2(u)du\Big)ds,$$

$$I_4(t) = -\frac{c(t)}{2}x^2(t) + \frac{1}{2}\int_0^t c'(s)x^2(s)ds,$$

and

$$\begin{aligned} L \;=\; & \Big(g'(x(0))x'(0)x''(0) + g(x(0))x'''(0)\Big)x(0) \\ & -g(x(0))x''(0)x'(0) + a(0)x(0)x''(0) \\ & -\frac{a(0)}{2}x'^2(0) + b(0)x(0)x'(0) + \frac{c(0)}{2}x^2(0). \end{aligned}$$

From (3), (14) and the conditions (i), (ii), (H1) and since $\lim\limits_{u\to 0}|g'(u)| \leq L_0 < \infty$, we have

$$\begin{aligned} I_1(t) \;\leq\; & \Big(|g'(x(t))||x'(t)x''(t)| + M|x'''(t)|\Big)|x(t)| + M|x''(t)x'(t)| + \\ & M\int_0^t x''^2(s)ds, \\ I_2(t) \;\leq\; & a_1|x''(t)x(t)| + \frac{a_1}{2}x'^2(t) + \frac{3}{2}D_3^2\int_0^t |a'(s)|ds, \end{aligned}$$

hence $\lim_{t\to+\infty} I_2(t) \leq \frac{3}{2}D^2\eta_1$,

$$\begin{aligned} I_3(t) \;\leq\; & b_1|x(t)x'(t)| + b_1\int_0^t x'^2(s)ds + (D_3^2 + L_1)\int_0^t |b'(s)|ds, \\ I_4(t) \;\leq\; & \frac{c_1}{2}x^2(t) + \frac{1}{2}D_3^2\int_0^t |c'(s)|ds, \end{aligned}$$

then $\lim_{t\to+\infty} I_4(t) \leq \frac{1}{2}D^2\eta_1$.
Therefore,

$$\lim_{t\to+\infty}\big(\sum_{n=1}^{4} I_n(t)\big) \leq ML_2 + \frac{3}{2}D^2\eta_1 + b_1D_2 + \eta_1(D^2 + L_1) + \frac{1}{2}D^2\eta_1 = L_4. \quad (21)$$

Consequently, from (20), (21) and condition (iii) we obtain

$$\int_0^\infty x^2(s)ds \le \frac{1}{d_0\delta}\int_0^\infty d(s)x(s)h(x(s))ds \le \frac{L_4+L}{d_0\delta}.$$

This completes the proof of Theorem.

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