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Motion planning algorithms, topological properties and affine approximation

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Abstract

The topological study of the so-called *motion planning algorithms* emerged in the 2003-2004 with the works of M. Farber (see [1], [2]). We focus here on the topological study of the set of these algorithms, when the configuration space is a normed vector space. We especially show that any motion planning algorithm in a compact sub-configuration space can be approximated by some piecewise affine ones.

Keywords: Motion planning algorithms, topological robotics, approximation, regular spaces.

1 Introduction

Michael Farber, motivated by the topological study of motion planning algorithms by using tools from algebraic topology, considered a path connected topological space X, and equipped its path space PX with the compact open topology. He viewed any motion planning algorithm of any mechanical robot that moves on the configuration space X to be an:

- Input: a pair (A, B) of two given points in the configuration space X;

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- Output: a continuous path from A to B and hence a continuous section

$$\begin{array}{rccl} s: & X \times X & \longrightarrow & PX \\ & (A,B) & \longmapsto & s(A,B) \end{array}$$

of the canonical projection

$$\begin{array}{rccl} \pi: & PX & \longrightarrow & X \times X \\ & \gamma & \longmapsto & (\gamma(0),\gamma(1)) \end{array} .$$

The path-connectedness of X ensures the existence of such section, its continuity (see Theorem 1, [1]) is equivalent that X must be contractible (i.e. homotopic to a point).

In the sequel, $\mathcal{M}(X)$ will denote the non empty set of such algorithms, where X is path-connected and contractible topological space. $\mathcal{M}(X)$ is topologized with the open-compact topology as a subset of $\mathrm{map}(X \times X, PX)$. Our first main result states that $\mathcal{M}(X)$ is contractible (Theorem 1). In other words, homotopically $\mathcal{M}(X)$ is trivial, but no necessary topologically poor. We will focus on the study of its topological properties by considering the special case, when X is a normed vector space.

In fact, when X is a normed vector space, the open-compact topology on PX coincides with that of the uniform convergence and this allows us to study the stability of a robot motion using norms. We will show that $\mathcal{M}(X)$ is regular (Theorem 2) and approximate any motion planning algorithm by a sequence of piecewise affine ones (Theorem 3). Many interpretations arise, for example from Theorem 2, one may conclude that in a normed configuration space, two robots can move by avoiding each other in two disjoint areas or that any robot can navigate outside a given compact area.

The paper is organized as follows: In section 2 we prove our mains results Theorems 1 and 2. In section 3, we focus on the affine approximation of any motion planning algorithm (MPA) on a compact sub-configuration space.

2 Motion planning algorithms: Topological properties

Theorem 1. *If X is a path-connected and contractible CW-complex, then $\mathcal{M}(X)$ is also contractible.*

Proof. Note that $\pi : PX \longrightarrow X \times X$ is a fibration in the sense of Hurewicz whose fiber is ΩX, the based loop space on X. Since X is contractible, then

π is a homotopy equivalence over $X \times X$, i.e. there is a section s of π and a homotopy $H : PX \times I \longrightarrow PX$ between the identity of PX and $s \circ \pi$ such that $\pi \circ H$ is a constant homotopy. Hence, the homotopy $(f,t) \mapsto H_t \circ f$ provides a contraction of $\mathcal{M}(X)$. □

Before proving our second main result, let us recall some basic topological definitions and fix some denotations that will be used on the sequel:

Definition 1 (see [3]). *A topological space X is a regular space if, given any nonempty closed set K and any point $x \notin K$, there exists a neighbourhood U of x and a neighbourhood V of K that are disjoint.*

Concisely speaking, it is always possible to separate x and K with disjoint neighborhoods, when $x \notin K$.

In what follows:

- $\mathcal{C}(X \times X, PX)$ denotes the set of continuous maps from $X \times X$ to PX;
- $\mathcal{C}_0(X \times X, PX)$ denotes that of continuous maps $f \in \mathcal{C}(X \times X, PX)$ satisfying the condition that: $f(A,B)(0) = f(A,B)(1)$ for any pair of points $(A,B) \in X \times X$.
- For any non empty subset K of $X \times X$, $\mathcal{M}_{K,PX}$ denotes the set of local motion planning algorithms on K, that are continuous sections $s : K \times K \longrightarrow PX$ of $\pi : PX \to X \times X$, (i.e., $\pi \circ s = \mathrm{id}_{K\times K}$);
- For any fixed compact subset K of $X \times X$, N_K denotes the semi-norm defined on $\mathcal{C}(X \times X, PX)$ by
$$N_K(f) = \sup_{(A,B)\in K} \|f(A,B)\|_\infty;$$
- For any $f \in \mathcal{C}(X \times X, PX)$ and $\varepsilon > 0$, $\mathbb{B}_K(f,\varepsilon)$ denotes the open semi-ball of $\mathcal{C}(X \times X, PX)$ defined by
$$B_K(f,\varepsilon) = \{g \in \mathcal{C}(X \times X, PX)/N_K(f-g) < \varepsilon\},$$
where $\|\gamma\|_\infty = \sup\limits_{t\in[0,1]} \|\gamma(t)\|$ is well defined for any $\gamma \in PX$.

We now endow $\mathcal{C}(X \times X, PX)$ with the topology of the compact uniform convergence, that is the topology induced by the family of the semi-norms $(N_K)_K$ and whose open sets are the disjoint unions of any finite intersections of semi-balls $\mathbb{B}_K(f,\varepsilon)$.

Theorem 2. *$\mathcal{M}(X)$ is a regular space.*

Proof. $\mathcal{C}(X \times X, PX)$ is a Hausdorff and locally convex topological vector space. Indeed, let $f \neq g$ in $\mathcal{C}(X \times X, PX)$ and consider $(x, y) \in X$ such that $f(x, y) \neq g(x, y)$. Since PX is Hausdorff , there exists $\varepsilon > 0$ such that $\mathbb{B}(f(x, y), \varepsilon) \cap \mathbb{B}(g(x, y), \varepsilon) = \emptyset$ and so $\mathbb{B}_K(f, \frac{\varepsilon}{2}) \cap \mathbb{B}_K(g, \frac{\varepsilon}{2}) = \emptyset$ where $K = \{(x, y)\}$. Therefore we conclude that $\mathcal{C}(X \times X, PX)$, $\mathcal{M}(X)$ and $\mathcal{C}_0(X \times X, PX)$ are regular spaces, since in Hausdorff and locally convex topological vector spaces, the unity admits a fundamental system of open and balanced neighbourhoods and a fundamental system of closed and balanced neighbourhoods (see [4]). □

3 Motion planning algorithms: Affine approximation.

Definition 2. *A motion planning algorithm s on X is called piecewise affine if and only if for any pair of points $A, B \in X$, there exists a subdivision $t_0 = 0 < t_1 < ... < t_{n-1} < t_n = 1$ of $[0, 1]$ such that*

$$s(A, B)(t) = a_i t + b_i$$

on each $[t_i, t_{i+1}]$.

Let $\mathcal{M}_X^{\mathrm{Aff}}$ denote the set of the piecewise affine motion planning algorithms on X and $\mathcal{M}_{K,PX}^{\mathrm{Aff}}$ denote that of local piecewise affine motion planning algorithms on any compact subset K of $X \times X$. Our main purpose is to prove that, when X is compact, any motion planning algorithms on X can be approximated by piecewise affine ones. This is very useful, in the sense that although we know enough about the existence of motion planning algorithms, we know less how to describe and determinate it explicitly, hence one can appeal to discretization methods.

For a fixed compact subset K of $X \times X$, we equip $\mathcal{M}_{K,PX}$ with the topology of locally convex spaces defined by the family of the semi-norms $N_{K \cap K'}$ where K' is an arbitrary compact subset of $X \times X$

Theorem 3. *If K is a compact subset of $X \times X$, then $\mathcal{M}_{K,PX}^{\mathrm{Aff}}$ is dense in $\mathcal{M}_{K,PX}$.*

Proof. Let $s : K \longrightarrow PX$ a local motion planning algorithm on K, $\varepsilon > 0$ and K' an arbitrary compact subset of $X \times X$. We are looking for a local motion planning algorithm $s' \in \mathcal{M}^{\text{Aff}}_{K\cap K',PX}$ such that $s' \in \mathbb{B}_{N_{K\cap K'}}(s,\varepsilon)$ (i.e., $\sup_{(A,B)\in(K\cap K')^2}(\|s(A,B)-s'(A,B)\|_\infty) < \varepsilon$). Indeed, the map

$$\begin{array}{rrcl} : & (K\cap K')^2\times[0,1] & \longrightarrow & X \\ & (A,B,t) & \longmapsto & s(A,B)(t) \end{array}$$

is continuous on the compact $(K\cap K')^2\times[0,1]$, hence is uniformly continuous. Therefore there exists $\eta > 0$ such that

$$\|(A,B,t)-(A',B',t')\|_{<}\eta \Longrightarrow \|s(A,B)(t)-s(A',B')(t'\| < \varepsilon.$$

Consider any integer n such that $\frac{1}{n} < \eta$, then subdivide $[0,1]$ with $\left(t_k = \frac{k}{n}\right)_{k=0..n}$ and define the following piecewise affine motion planning algorithm:

$$\begin{array}{l} s_n(A,B)(t) = (n(t_i-t)+1)s(A,B)(t_i) + n(t-t_i)s(A,B)(t_{i+1}); \;\; t\in]t_i,t_{i+1}[\\ s_n(A,B)(t_i) = s(A,B)(t_i). \end{array}$$

We have $\|s(A,B)(t)-s_n(A,B)(t)\| = \|\,(n(t_i-t)+1)\,(s(A,B)(t)-s(A,B)(t_i))+ n(t_i-t)\,(s(A,B)(t)-s(A,B)(t_{i+1}))\,\| \leq (n(t_i-t)+1)\|s(A,B)(t)-s(A,B)(t_i)\|+ n(t_i-t)\|s(A,B)(t)-s(A,B)(t_{i+1})\| < (n(t_i-t)+1)\varepsilon + n(t_i-t)\varepsilon = \varepsilon.$ □

Remarks:

- It is easy to check that Theorem 3 holds for any finite intersection of semi-balls $\mathbb{B}_{N_{K\cap K'}}(s,\varepsilon)$.
- We can also investigate in Theorem 3 what can be said about the approximation of motion planning algorithm by piecewise polynomial ones such B-Splines, Bézier curves, …(see [5]).
- In Hilbert spaces, the results of Theorem 3 are still available, but other ones may arise by using the basic properties of projections or that of least squares.

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