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Loop topological complexity

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Abstract

We introduce here the notion of loop motion planning algorithms and show that it yields to a homotopical invariant: the loop topological complexity, denoted throughout this paper by $\mathrm{TC}^{\mathrm{LP}}(-)$, which measures the algorithmic complexity of the motion of a drone as, for example, an unmanned airplane or a guided TV camera. Our main result states that $\mathrm{TC}(-) = \mathrm{TC}^{\mathrm{LP}}(-)$, where TC denotes the ordinary topological complexity introduced by M. Farber. Some interesting applications will emerge and will be discussed.

Keywords: Motion planning algorithm, topological robotics, topological complexity, loop topological complexity, monoidal topological complexity, Iwase-Sakai conjecture.

1 Introduction

In the framework of topological robotics, spaces X are assumed to be path-connected and are viewed as some configuration spaces of all the states of a given mechanical system (a robot for example). M. Farber defined a *motion planning algorithm* to be any continuous section $s : X \times X \longrightarrow PX$ of the bi-evaluation

$$\begin{array}{rccc} ev : & PX & \longrightarrow & X \times X \\ & \gamma & \longmapsto & (\gamma(0), \gamma(1)) \end{array} .$$

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Here, the *free path space*, $PX := \{\gamma : [0,1] \longrightarrow X \text{ continuous}\}$ is endowed with the open-compact topology. The existence of such algorithms is insured by the path-connectedness of X, while their continuity is equivalent to the contractibility of X. (Theorem 1, [1]).

Roughly speaking, the continuity of motion planners means that close initial-final pairs (A, B) and (A', B') produce close motions $s(A, B)$ and $s(A', B')$. In other words it interprets the stability of the motion. In order to measure the complexity of this stability, M. Farber defined the homotopy invariant TC(X), named *topological complexity*, to be the minimum among the normalized cardinalities of all open coverings $(U_i)_{0\leqslant i\leqslant k}$ of $X \times X$, over each of which ev has a local continuous section $s_i : U_i \longrightarrow PX$ (i.e., $ev \circ s_i = \mathrm{id}_{\mathrm{U_i}}$). For example, we have:

- $\mathrm{TC(X)} = 0$ if and only if X is contractible (Theorem 1, [1]);
- $\mathrm{TC(X)} = 1$ if and only if X is homotopy equivalent to some odd-dimensional sphere (Corollary 1, [2]);
- $\mathrm{TC}(\mathbb{S}^{\mathrm{n}}) = 2$, whenever n is even (Theorem 8, [1]).

In his founding paper ([1]), M. Farber was not interested about the return motion. He makes up for some years after and focused with M. Grant on the symmetric case (i.e., when the robot's goings and comings motions are the same). They defined the notion of *symmetric topological complexity*, denoted $\mathrm{TC^S}(-)$, and showed that

$$\mathrm{TC(X)} \leqslant \mathrm{TC^S(X)} \quad \text{Theorem 9, [3]).} \tag{1}$$

One may be attempted to waive this restriction on the return motion and let the robot free to take any arbitrary way to come back to its departure point. That was our first inspiration to define and study topologically and homotopically the concept of *loop motion planning algorithm* (LMPA for short).

2 Loop motion planning algorithms

Definition 1. *A LMPA over X, is any continuous section $s : X \times X \longrightarrow LX$ of the loop bi-evaluation*

$$\begin{array}{rcccl} ev^{\mathrm{LP}} & : & LX & \longrightarrow & X \times X \\ & & \gamma & \longmapsto & \left(\gamma(0), \gamma\left(\tfrac{1}{2}\right)\right) \end{array}.$$

Here, the *free loop space*, $LX := \{\gamma : X^{S^1} \longrightarrow X \text{ continuous}\}$ is endowed with open-compact topology, the input is a pair of points (A=departure, B=target), while the output should suggest to the robot a target by requiring a come-back to the departure point. The motion of a drone like an unmanned airplane or a guided TV camera can find many interesting interpretations here. The famous NP-complete problem of vehicle routing with pick-up and delivery can also be studied throughout this angle. We first prove that :

Theorem 1. *If X is a path-connected topological space, then LMPAs on X exist if and only if X is contractible.*

Proof. Let $s : X \times X \longrightarrow LX$ be a LMPA (i.e., $ev^{\mathrm{LP}} \circ s = \mathrm{id}_{\mathrm{X\times X}}$). In other words

$$s(A,B)(0) = s(A,B)(1) = A \text{ and } s(A,B)\,(1/2) = B$$

for any pair $(A,B) \in X \times X$. Fix now an arbitrary point A_0 in X and consider the homotopy

$$\begin{array}{rcccl} H & : & X \times [0,1] & \longrightarrow & X \\ & & (A,t) & \longmapsto & s(A_0,A)\left(\frac{t}{2}\right) \end{array}.$$

We have $H(A,0) = A_0$ and $H(A,1) = A$. Thus H contracts the whole space X into the given point A_0 (i.e., X is contractible).

Conversely, suppose that X is contractible. Then, fix $A_0 \in X$ and consider a homotopy $H : X \times [0,1] \longrightarrow X$ such that $H(A,0) = A_0$ and that $H(A,1) = A$ for any $A \in X$. Let $(A,B) \in X \times X$ and set

$$s(A,B)(t) = \begin{cases} H(A, 1-4t) & \text{if} \quad 0 \leqslant t \leqslant \frac{1}{4} \\ H(B, 4t-1) & \text{if} \quad \frac{1}{4} \leqslant t \leqslant \frac{1}{2} \\ H(B, 3-4t) & \text{if} \quad \frac{1}{2} \leqslant t \leqslant \frac{3}{4} \\ H(B, 4t-3) & \text{if} \quad \frac{3}{4} \leqslant t \leqslant 1 \end{cases}$$

It is clear that s is continuous, that $s(A,B)(0) = s(A,B)(1) = A$ and that $s(A,B)\,(1/2) = B$. In other words that s is a LMPA. ☐

Following Farber's spirit we define the *loop topological complexity* of X, denoted here $\mathrm{TC}^{\mathrm{LP}}(\mathrm{X})$, to be :

Definition 2. *The minimum among the normalized cardinalities of all open coverings $(U_i)_{0 \leqslant i \leqslant k}$ of $X \times X$, over each of which ev^{LP} has a local continuous section $s_i : U_i \longrightarrow LX$ (i.e., $ev^{\mathrm{LP}} \circ s_i = \mathrm{id}_{\mathrm{U_i}}$).*

In particular, we have :

Theorem 2.

$$\mathrm{TC}^{\mathrm{LP}}(\mathrm{X}) = \mathrm{TC}(\mathrm{X}).$$

Proof. Suppose one has an open set U in $X \times X$ and a map $s : U \longrightarrow LX$ such that $s(A,B)(0) = A, s(A,B)(1/2) = B$ and $s(A,B)(1) = A$. Then if we set $z(A,B)(t) = s(A,B)(t/2)$, we get that $z : U \longrightarrow PX$ is a section of the free path fibration ev. So $\mathrm{TC}(\mathrm{X}) \leqslant \mathrm{TC}^{\mathrm{LP}}(\mathrm{X})$.

On the other hand, if $z : U \longrightarrow PX$ is a section of the free path fibration ev, then we can set $s(A,B)(t) = z(A,B)(2t)$ for $0 \leqslant t \leqslant 1/2$ and $s(A,B)(t) = z(A,B)(2-2t)$ for $1/2 \leqslant t \leqslant 1$. Then $s : U \longrightarrow LX$ satisfies the conditions to be a loop motion planner. So $\mathrm{TC}^{\mathrm{LP}}(\mathrm{X}) \leqslant \mathrm{TC}(\mathrm{X})$. □

Theorem 3. $\mathrm{TC}^{\mathrm{LP}}(-)$ *is a homotopy invariant.*

Proof. Obvious from Theorem 2, since TC is a homotopy invariant (Theorem 3, [1]). □

3 Applications

As mentionned here above, our results yield to many applications and connections with other results or well known open problems. The most awesome, but not the least one, is that our loop TC generalizes both the classical notion (introduced by M. Farber in [1]) and the symmetric one (introduced by M. Farber and M. Grant in [3]). Moreover, whilst the symmetry of a goings and comings robot motion increases the complexity navigation (see inequality (1)), our requirement that the return is free does not (Theorem 2).

Note also the deep intersection between the loop TC and the monoidal TC (denoted $\mathrm{TC}^{\mathrm{M}}(-)$ and introduced by Iwase-Sakai in [4]): the two ones focus on the special case when the initial position of a robot motion coincides with the terminal one. Theorem 2 may be viewed as a possible research direction supporting the famous Iwase-Sakai conjecture (see [5]) about coincidence of the ordinary topological complexity TC and the monoidal one TC^{M}.

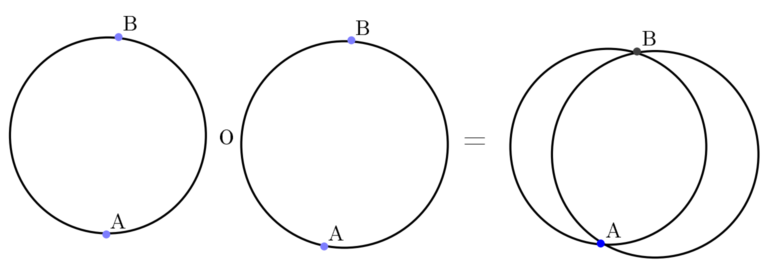
It is also worth to point out that our notion of LMPA extends, at level of free loops $\gamma \in LX$, the notion of *midpoint maps*, introduced initially by M. Farber and M. Grant in [3] at level of free loops $\gamma \in PX$.

As it is done in [6], one may be attempted to study the topological, homotopical or algebraic behaviour of the set of LMPA, denoted here $\mathcal{M}_{\mathrm{LP}}(X)$. First, $\mathcal{M}_{\mathrm{LP}}(X)$ as a map space can be topologized with the induced open compact topology. Secondly, Theorem 1 states that $\mathcal{M}_{\mathrm{LP}}(X)$ is non empty if and only if X is contractible. In this case (see Lemma 1, [6]), $\mathcal{M}_{\mathrm{LP}}(X)$ is also contractible.

Finally, for any given LMPAs s_1 and s_2 we (inspired from the natural loop concatenation) define their *loop motion product* as follows :

$$\begin{aligned} s_1 \circ s_2(A,B)(t) &= s_1(A,B)(t) && \text{if } 0 \leqslant t \leqslant \tfrac{1}{2} \\ &= s_1(A,B)(3t-1) && \text{if } \tfrac{1}{2} \leqslant t \leqslant \tfrac{2}{3} \\ &= s_2(A,B)(3t-2) && \text{if } \tfrac{2}{3} \leqslant t \leqslant 1 \end{aligned}$$

As illustrated here bellow, two LMPAs are composable if and only if they have two common base points.



Open question : Though $\mathcal{M}_{\mathrm{LP}}(X)$ is homotopically trivial, algebraically it is not. The natural questions are: What can one do or interpret (in terms of robotics) with this loop motion product? What structure does it induce?

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