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$L(2,1)$-Labeling for Subdivisions of Some Cycle Dominated Graphs

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Abstract

Let $G(V,E)$ be a simple, finite, connected, undirected graph. Distance two labeling or $L(2,1)$-labeling of a graph G is an assignment f from the vertex set $V(G)$ to the set of non-negative integers such that $|f(x) - f(y)| \geq 2$ if x and y are adjacent and $|f(x) - f(y)| \geq 1$ if x and y are at distance 2, for all x and y in $V(G)$. The $L(2,1)$-labeling number $\lambda(G)$ of G is the smallest number k such that G has an $L(2,1)$-labeling f with $\max\{f(v) : v \in V(G)\} = k$. In this paper, we construct $L(2,1)$-labeling of subdivisions of cycle dominated graphs like subdivided Double Fans, subdivided nC_α with a common vertex and subdivided Books B_n and hence we find the λ-number of these graphs.

Keywords: Distance two labeling, transmitters, Channel assignment, Double Fan.

1 Introduction

For standard terminology and notation, we follow Bondy and Murty [1] or Murugan [2]. We give a brief summary of concepts and definitions which serves as prerequisites to this paper. Unless or otherwise mentioned, let $G(V,E)$ be a simple, finite, connected, undirected graph without loops or multiple edges.

Following the standard terminology, we use P_n to denote a path on n vertices, C_n to denote a cycle on n vertices, B_n to denote a Book with n

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pages, Δ to denote the maximum degree of a graph, $\lceil x \rceil$ to denote the least integer greater than or equal to x, and $\lfloor x \rfloor$ to denote the greatest integer less than or equal to x.

A labeling of a graph is a function that sends some set of graph element to a set of positive integers. If the domain is $V(G)$ or $E(G)$ or $V(G) \cup E(G)$, then the labeling is called vertex labeling or edge labeling or total labeling respectively. The edge-weight of an edge uv under a vertex labeling is the sum of the vertex labels at its ends; under a total labeling, we also add the label of uv.

Definition 1. *Two graphs G_1 and G_2 are said to be disjoint if they have no vertex in common and edge-disjoint if they have no edge in common.*

Definition 2. *The union of two graphs G_1 and G_2 is another graph G_3 with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. We denote n pairwise disjoint copies of G as nG.*

Definition 3. *If G_1 and G_2 are disjoint graphs, then the join of G_1 and G_2, written as $G_1 + G_2$, is the graph consisting of the union $G_1 \cup G_2$, together with all edges of the type v_1v_2, where $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$.*

Definition 4. *An edge e is said to be subdivided when it is deleted and replaced by a path of length two connecting its ends, the internal vertex being a new vertex.*

In the next section, we provide a brief introduction about $L(2,1)$-labeling or Distance two labeling and some existing results in this area.

2 $L(2,1)$-labeling and some existing results

Many authors ([3],[4] and [5]) have contributed to the study of "T-colorings" of graphs, which is motivated by the task of assigning channel frequency without interference. The unprecedented growth of wireless communication made these studies more popular. This is nothing but the channel assignment problem. The original notion of distance two labeling can be seen in the context of frequency assignment, where 'close' transmitters must receive different frequencies and 'very close' transmitters must receive frequencies that are at least two frequencies apart so that they can avoid interference. The mathematical abstraction of this concept is $L(2,1)$-labeling.

An $L(2,1)$-labeling of a graph G is an assignment f from the vertex set $V(G)$ to the set of non-negative integers such that $|f(x)-f(y)| \geq 2$ if x and y are adjacent and $|f(x)-f(y)| \geq 1$ if x and y are at distance 2, for all x and y in $V(G)$. A k-$L(2,1)$-labeling is an $L(2,1)$-labeling $f : V(G) \to \{0,1,2,\ldots,k\}$, and we are interested to find the minimum k among all possible assignments. This invariant, the minimum k, is known as the $L(2,1)$-labeling number or λ-number and is denoted by $\lambda(G)$. The generalization of this concept is as below.

For positive integers k, d_1, d_2, a k-$L(d_1,d_2)$-labeling of a graph G is a function $f : V(G) \to \{0,1,2,\ldots,k\}$ such that $|f(u) - f(v)| \geq d_i$ whenever the distance between u and v in G, $d_G(u,v) = i$, for $i = 1, 2$. The $L(d_1,d_2)$-number of G, $\lambda_{d_1,d_2}(G)$, is the smallest k such that there exists a k-$L(d_1,d_2)$-labeling of G. Many results on trees are available in this area and strict bounds of λ are obtained for trees. Therefore, we are interested in cycle dominated graphs. Murugan [6] has obtained λ-numbers for some cycle dominated graphs. In this paper, we focus on subdivisions of some cycle dominated graphs, using recursion.

- In [7], Griggs and Yeh have discussed $L(2,1)$-labeling for path, cycle, tree and cube. They have derived results for the graphs of diameter 2. They have shown that the $\lambda(T)$ for trees with maximum degree $\Delta \geq 1$ is either $\Delta + 1$ or $\Delta + 2$.

- Chang and Kuo [8] provided an algorithm to obtain $\lambda(T)$.

- Vaidya and Bantava[9] have discussed $L(2,1)$-labeling of cacti.

- Vaidya et.al. [10] have discussed $L(2,1)$-labeling in the context of some graph operations.

- Yeh [11] have discussed the $L(2,1)$-labeling on various class of graphs like trees, cycles, chordal graphs, Cartesian products of graphs etc.,

- Griggs and Yeh [7] proved that if a graph G contains three vertices of degree Δ such that one of them is adjacent to the other two, then $\lambda(G) \geq \Delta + 2$, where Δ is the maximum degree of G.

- Griggs and Yeh [7] posed a conjecture that $\lambda(G) \leq \Delta^2$ for any graph with $\Delta \geq 2$, where Δ is the maximum degree of G, and they proved that $\lambda(G) \leq \Delta^2 + 2\Delta$ at the same time.

- Chang and Kuo [12] proved that $\lambda(G) \leq \Delta^2 + \Delta$, for any graph with $\Delta \geq 2$, where Δ is the maximum degree of G.
- Kral and Skrekovski [13] proved that $\lambda(G) \leq \Delta^2 + \Delta - 1$, for any graph with $\Delta \geq 2$, where Δ is the maximum degree of G.
- Goncalves [14] proved that $\lambda(G) \leq \Delta^2 + 2\Delta - 2$, for any graph with $\Delta \geq 2$, where Δ is the maximum degree of G.

Definition 5. *A fan F_m is the join of P_m with K_1. That is, $F_m = P_m + K_1$.*

If the vertices of P_m are $v_1, v_2, \ldots, v_m$ and u represents K_1 then the path P_m is called the base line of the fan F_m and uv_i, $i = 1, 2, \ldots, m$ are called the supporting lines of the fan F_m.

Here, the subdivision of a fan F_m is the edge subdivision of the edges of F_m such that all supporting lines have the same number of vertices.

Definition 6. *A double fan is the join of P_m with $2K_1$.*

3 Results

Theorem 1. *The λ-number of a subdivided double fan with base line P_n is $n+1$, $n \geq 9$.*

Proof. Let G be the subdivided double fan with base line P_n. Consider the path P_n. Starting with the initial vertex of P_n, name the alternate vertices as $v_1, v_2, \ldots, v_{\lceil n/2 \rceil}$ and continue naming the remaining vertices of P_n starting with the second vertex as $v_{\lceil n/2 \rceil+1}, v_{\lceil n/2 \rceil+2}, \ldots, v_n$. Name the two isolated vertices of $2K_1$ as u_1 and u_2. Apart from the path P_n, the other mutually internally disjoint paths of G are $v_i, v_{i,1}, v_{i,2}, v_{i,3}, \ldots, v_{i,k}, v_{i,l}, u_1$, $i = 1, 2, \ldots, n$ and $v_i, u_{i,1}, u_{i,2}, u_{i,3}, \ldots, u_{i,k}, u_{i,l}, u_2$, $i = 1, 2, \ldots, n$.

Define $f : V(G) \to \{0, 1, 2, \ldots, n+1\}$ as follows.

$f(u_1) = f(u_2) = 0$,

For $i = 1, 2, \ldots, n$,

$$\begin{aligned} f(v_i) &= i+1, \\ f(v_{i,1}) &= f(v_i) + 2 \quad \text{if } f(v_i) + 2 \leq n+1. \\ &= 2, \quad \text{otherwise.} \end{aligned}$$

For $i = 1$, define

$$\begin{aligned} f'(v_{1,2}) &= f(v_{1,1}) + 2 \text{ if } f(v_{1,1}) + 2 \neq f(v_{\lceil n/2 \rceil + 1}). \\ &= f(v_{1,1}) + 4, \text{ otherwise.} \end{aligned}$$

For $i = 2, 3, \ldots, \lceil n/2 \rceil$, define

$$\begin{aligned} f'(v_{i,2}) &= f(v_{i,1}) + 2 \text{ if } f(v_{i,1}) + 2 \neq f(v_{\lceil n/2 \rceil + i - 1}) \text{ and } f(v_{\lceil n/2 \rceil + i}). \\ &= f(v_{i,1}) + 4, \text{ otherwise.} \end{aligned}$$

For $i = \lceil n/2 \rceil + 1, \lceil n/2 \rceil + 2, \ldots, n$, define

$$\begin{aligned} f'(v_{i,2}) &= f(v_{i,1}) + 2 \text{ if } f(v_{i,1}) + 2 \neq f(v_{i - \lceil n/2 \rceil}) \text{ and } f(v_{i - \lceil n/2 \rceil + 1}). \\ &= f(v_{i,1}) + 4, \text{ otherwise.} \end{aligned}$$

Now, for $i = 1, 2, \ldots, n$, we define

$$\begin{aligned} f(v_{i,2}) &= f'(v_{i,2}) \text{ if } f'(v_{i,2}) \leq n + 1 \\ &= 4 \text{ if } f'(v_{i,2}) > n + 1 \text{ and } i = \lceil n/2 \rceil + 1. \\ &= 2 \text{ if } f'(v_{i,2}) > n + 1 \text{ and } i \neq \lceil n/2 \rceil + 1. \end{aligned}$$

For $i = 1$ define

$$\begin{aligned} f'(v_{1,3}) &= f(v_{1,2}) + 2 \text{ if } f(v_{1,2}) + 2 \neq f(v_{\lceil n/2 \rceil} + 1) \\ &= f(v_{1,2}) + 4, \text{ otherwise.} \end{aligned}$$

For $i = 2, 3, \ldots, \lceil n/2 \rceil$, define

$$\begin{aligned} f'(v_{i,3}) &= f(v_{i,2}) + 2 \text{ if } f(v_{i,2}) + 2 \neq f(v_{\lceil n/2 \rceil + i - 1}) \text{ and } f(v_{\lceil n/2 \rceil + i}). \\ &= f(v_{i,2}) + 4, \text{ otherwise.} \end{aligned}$$

For $i = \lceil n/2 \rceil + 1, \lceil n/2 \rceil + 2, \ldots, n$, define

$$\begin{aligned} f'(v_{i,3}) &= f(v_{i,2}) + 2 \text{ if } f(v_{i,2}) + 2 \neq f(v_{i - \lceil n/2 \rceil}) \text{ and } f(v_{i - \lceil n/2 \rceil + 1}). \\ &= f(v_{i,2}) + 4, \text{ otherwise.} \end{aligned}$$

Now, for $i = 1, 2, \ldots, n$, we define

$$\begin{aligned} f(v_{i,3}) &= f'(v_{i,3}) \quad \text{if } f'(v_{i,3}) \leq n + 1 \\ &= 4 \quad \text{if } f'(v_{i,3}) > n + 1 \text{ and } i = \lceil n/2 \rceil + 1. \\ &= 2 \quad \text{if } f'(v_{i,3}) > n + 1 \text{ and } i \neq \lceil n/2 \rceil + 1. \end{aligned}$$

We use these labels to label the other vertices $v_{i,j}$, $i = 1, 2, \ldots, n$ and $j = 4, 5, \ldots, k$.

If $j \equiv p(\text{mod } 3)$, define

$$\begin{aligned} f(v_{i,j}) &= f(v_{i,1}) \text{ if } p = 1 \\ &= f(v_{i,2}) \text{ if } p = 2 \\ &= f(v_{i,3}) \text{ if } p = 0 \end{aligned}$$

Next, define $f(v_{i,l}) = i + 1$, $i = 1, 2, \ldots, n$.

Now, we label the vertices $u_{i,j}, i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, k$.

For $i = 1, 2, \ldots, n$, define

$$f(u_{i,1}) = f(v_{i,2}), f(u_{i,2}) = f(v_{i,3}) \text{ and } f(u_{i,3}) = f(v_{i,1}).$$

We use these labels to label the other vertices $u_{i,j}, i = 1, 2, \ldots, n$ and $j = 4, 5, \ldots, k$. If $j \equiv p(\text{mod } 3)$, define

$$\begin{aligned} f(u_{i,j}) &= f(u_{i,1}) \text{ if } p = 1 \\ &= f(u_{i,2}) \text{ if } p = 2 \\ &= f(u_{i,3}) \text{ if } p = 0 \end{aligned}$$

Next, define $f(u_{i,l}) = i + 1$, $i = 1, 2, \ldots, n$.

Now we prove that f is distance two labeling.

Since the alternate vertices of P_n, starting with the initial vertex, are named as $v_1, v_2, \ldots, v_{\lceil n/2 \rceil}$ and the remaining vertices of P_n, starting with the second vertex, are named as $v_{\lceil n/2 \rceil+1}, v_{\lceil n/2 \rceil+2}, \ldots, v_n$, and $f(v_i) = i + 1$, $i = 1, 2, \ldots, n$ both the conditions of distance two labeling are satisfied for the vertices of P_n.

We note that $d(u_1, v_{i,l}) = 1$, $i = 1, 2, \ldots, n$. Since $f(u_1) = 0$, $f(v_{i,l}) = i+1$, $i = 1, 2, \ldots, n$, $|f(u_1) - f(v_{i,l})| \geq 2$. Also $d(v_{i,l}, v_{i+1,k}) = 2$, $i = 1, 2, \ldots, n-1$. By construction of f, $|f(v_{i,l}) - f(v_{i+1,k})| \geq 1$. Clearly, the labels of the vertices of all internally disjoint (u_1, v_i)-paths satisfies both the conditions of distance two labeling.

We note that $d(u_2, u_{i,l}) = 1$, $i = 1, 2, \ldots, n$. Since $f(u_2) = 0$, $f(u_{i,l}) = i + 1$, $i = 1, 2, \ldots, n$, $|f(u_2) - f(u_{i,l})| \geq 2$. Also $d(u_{i,l}, u_{i+1,k}) = 2$, $i = 1, 2, \ldots, n-1$. By construction of f, $|f(u_{i,l}) - f(u_{i+1,k})| \geq 1$. Clearly, the labels of the vertices of all internally disjoint (u_2, v_i)-paths satisfies both the conditions of distance two labeling.

For all other vertices, both the conditions of distance two labeling are immediate. Hence f is a $L(2,1)$-Labeling. Therefore, $\lambda(G) \leq n + 1$. Since the maximum degree of G is n, $\lambda(G) \geq n + 1$. Hence $\lambda(G) = n + 1$. □

Definition 7. *A Book B_n is the product of the star $K_{1,n}$ with K_2.*

A Book is nothing but a collection of C_4's with a common edge. A Book B_n has $2n+2$ vertices, $3n+1$ edges, n pages and maximum degree $n+1$. Now subdivide the edges of B_n, except the common edge, such that each page has the same number of vertices after subdivisions. So, a Book after edge subdivisions is nC_α with a common edge, where the number of vertices of each page of the Book is α.

Theorem 2. *The λ-number of a subdivided Book, nC_α with a common edge is $n+2$, $n \geq 6$.*

Proof. Let G be a subdivided Book, nC_α with a common edge. It has n pages, α vertices on each page and maximum degree $n+1$.

Let the vertices on the i-th page be $u, u_{i,1}, u_{i,2}, \ldots, u_{i,k}$ and v, where u and v lie on the common edge uv, $i = 1, 2, \ldots, n$ where $k+2 = \alpha$.

Clearly $V(G) = \{u, v, u_{i,j}/1 \leq i \leq n, 1 \leq j \leq k\}$.

Case 1: n odd.

Define $f : V(G) \to \{0, 1, 2, \ldots, n+2\}$ as follows.

$$\begin{aligned} f(u) &= 0 \\ f(v) &= n+2 \end{aligned}$$

First, we consider the vertices $u_{i,j}$ of even pages, that is, $i = 2, 4, \ldots, n-1$ and $j = 1, 2, \ldots, k$.

For $i = 2$,

If $j \equiv p(\mathrm{mod}(n+1)/2)$ then,

$$\begin{aligned} f(u_{2,p}) &= 2p+1 \quad \text{if } p = 1, 2, 3, \ldots, \tfrac{n+1}{2} - 1 \\ &= 1 \quad\quad\;\; \text{if } p = 0 \end{aligned}$$

For $i = 4, 6, \ldots, n-1$

$$\begin{aligned} f(u_{i,p}) &= f(u_{i-2,p}) + 2 \quad \text{if} \quad f(u_{i-2,p}) \leq n-2 \\ &= 1 \quad\quad\quad\quad\quad\quad\quad \text{otherwise} \end{aligned}$$

Now, we consider the vertices $u_{i,j}$ of odd pages, that is, $i = 1, 3, \ldots, n-2, n$ and $j = 1, 2, \ldots, k$.

For $i = 1$,

If $j \equiv p \bmod ((n-1)/2)$ then,

$$\begin{aligned} f(u_{1,p}) &= 2p \quad\quad\; \text{if } p = 1, 2, 3, \ldots, \tfrac{n-1}{2} - 1 \\ &= n-1 \quad \text{if } p = 0 \end{aligned}$$

For $i = 3, 5, \ldots, n-2$

$$\begin{aligned} f(u_{i,p}) &= f(u_{i-2,p}) + 2 \quad \text{if} \quad f(u_{i-2,p}) \leq n-3 \\ &= 2 \quad \text{otherwise} \end{aligned}$$

For $i = n$ and $j = 1$,

$$f(u_{n,j}) = f(u_{n-1,j}) + 1$$

For $i = n$ and $j = 2, 3, \ldots, k$,

$$\begin{aligned} f(u_{n,j}) &= f(u_{n-1,j}) + 2 \quad \text{if } f(u_{n-1,j}) \leq n-2 \\ &= 1 \quad \text{otherwise} \end{aligned}$$

Case 2: n even.
Define $f : V(G) \to \{0, 1, 2, \ldots, n+2\}$ as follows.

$$\begin{aligned} f(u) &= 0 \\ f(v) &= n+2 \end{aligned}$$

First, we consider the vertices $u_{i,j}$ of odd pages, that is, $i = 1, 3, \ldots, n-1$ and $j = 1, 2, \ldots, k$. For $i = 1$,
If $j \equiv p \mod (n/2)$ then,

$$\begin{aligned} f(u_{1,p}) &= 2p \quad \text{if } p = 1, 2, 3, \ldots, \tfrac{n}{2} - 1 \\ &= n \quad \text{if } p = 0 \end{aligned}$$

For $i = 3, 5, \ldots, n-1$

$$\begin{aligned} f(u_{i,p}) &= f(u_{i-2,p}) + 2 \quad \text{if} \quad f(u_{i-2,p}) \leq n-2 \\ &= 2 \quad \text{otherwise} \end{aligned}$$

Now, we consider the vertices $u_{i,j}$ of even pages, that is, $i = 2, 4, \ldots, n-2, n$ and $j = 1, 2, \ldots, k$.
For $i = 2$,
If $j \equiv p (\mod n/2)$ then,

$$\begin{aligned} f(u_{2,p}) &= 2p+1 \quad \text{if } p = 1, 2, 3, \ldots, \tfrac{n}{2} - 1 \\ &= 1 \quad \text{if } p = 0 \end{aligned}$$

For $i = 4, 6, \ldots, n-2$

$$\begin{aligned} f(u_{i,p}) &= f(u_{i-2,p}) + 2 \quad \text{if} \quad f(u_{i-2,p}) \leq n-3 \\ &= 1 \quad \text{otherwise} \end{aligned}$$

For $i = n$ and $j = 1$,

$$f(u_{n,j}) = f(u_{n-2,j}) + 2$$

For $i = n$ and $j = 2, 3, \ldots, k$,

$$\begin{aligned} f(u_{n,j}) &= f(u_{n-2,j}) + 2 \quad \text{if } f(u_{n-2,j}) \leq n-3 \\ &= 1 \quad \text{otherwise} \end{aligned}$$

Now we prove that f is $L(2,1)$-Labeling.
We note that $d(u,v) = d(u,u_{i,1}) = d(v,u_{i,k}) = 1$, $i = 1, 2, \ldots, n$.
Also $d(u_{i,j}, u_{i,j+1}) = 1$, $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, k-1$.
We note that $f(u) = 0$, $f(v) = n+2$ and $f(u_{i,1}) = i+1 \geq 2$, $i = 1, 2, \ldots, n$.
Now, $|f(u) - f(v)| = n+2 \geq 2$, $|f(u) - f(u_{i,1})| = i+1 \geq 2$, $i = 1, 2, \ldots, n$.
Since $f(u_{i,j}) \leq n$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, k$, except for $f(u_{n,1})$ and $f(v) = n+2$, we have $|f(v) - f(u_{i,k})| \geq 2$, $i = 1, 2, \ldots, n$.

When n is odd, the vertices $u_{1,j}$, $j = 1, 2, \ldots, k$ receive colours $2, 4, \ldots, n-1$ cyclically and when n is even, the vertices $u_{1,j}$, $j = 1, 2, \ldots, k$ receive colours $2, 4, \ldots, n$ cyclically and so in both the cases, we have $|f(u_{1,j}) - f(u_{1,j+1})| \geq 2$, $j = 1, 2, \ldots, k-1$.

When n is odd, the vertices $u_{2,j}$, $j = 1, 2, \ldots, k$ receive colours $3, 5, \ldots, n, 1$ cyclically and when n is even, the vertices $u_{2,j}$, $j = 1, 2, \ldots, k$ receive colours $3, 5, 7, \ldots, n-1, 1$ cyclically and so in both the cases, we have

$$|f(u_{2,j}) - f(u_{2,j+1})| \geq 2, \quad j = 1, 2, \ldots, k-1.$$

Hence, by the method of construction of f,

$$|f(u_{i,j}) - f(u_{i,j+1})| \geq 2, for \quad i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, k-1.$$

Now we consider vertices at a distance 2. First we note that for $i \neq \alpha$, $d(u_{i,1}, u_{\alpha}, 1) = 2$. Since $f(u_{i,1}) = i+1$, $i = 1, 2, \ldots, n$, we have

$$|f(u_{i,1}) - f(u_{\alpha,1})| \geq 1.$$

Now we consider the vertices $u_{i,k}$, $i = 1, 2, \ldots, n$ such that $d(u_{i,k}, u_{\alpha,k}) = 2$, $i \neq \alpha$. Since $f(u_{i,1})$ are all distinct, $f(u_{i,k})$ are all distinct by construction of f and so $|f(u_{i,k}) - f(u_{\alpha,k})| \geq 1$, $i \neq \alpha$.

For all other vertices which are at a distance 2, the absolute value of their label difference is greater than or equal to 1 is immediate. Hence f is a $L(2,1)$-Labeling. Therefore, $\lambda(G) \leq n+2$.

Since the maximum degree of G is $n+1$, $\lambda(G) \geq n+2$. Hence $\lambda(G) = n+2$. □

From theorem 2, the following algorithm is evident.

Algorithm 3:

Input : A subdivided Book, $nC\alpha$ with a common edge.

Output : Distance two labeling of the subdivided Book.

Step 1: Let the vertices of the n-th page be $u, u_{i,1}, u_{i,2}, u_{i,3}, \ldots, u_{i,k}, v$ where u and v lie on the common edge uv.

Step 2: Label u with the label 0 and v with the label $n+2$.

Step 3: If n is odd,

(i) Label the vertices of the first page $u_{1,1}, u_{1,2}, u_{1,3}, \ldots, u_{1,k}$ with the labels $2, 4, 6, \ldots, n-1$ cyclically.

(ii) Label the vertices of the i-th page, $i = 3, 5, \ldots, n-2$ by

$$\begin{aligned} f(u_{i,p}) &= f(u_{i-2,p}) + 2 \quad \text{if } f(u_{i-2,p}) \leq n-3 \\ &= 2 \quad \text{otherwise} \end{aligned}$$

(iii) Label the vertices of the second page $u_{2,1}, u_{2,2}, u_{2,3}, \ldots, u_{2,k}$ with the labels $3, 5, 7, \ldots, n, 1$ cyclically.

(iv) Label the vertices of the i-th page, $i = 4, 6, \ldots, n-1$ by

$$\begin{aligned} f(u_{i,p}) &= f(u_{i-2,p}) + 2 \quad \text{if } f(u_{i-2,p}) \leq n-2 \\ &= 1 \quad \text{otherwise} \end{aligned}$$

(v) Label the vertices of the n-th page by,
For $i = n$ and $j = 1$, $f(u_{n,j}) = f(u_{n-1,j}) + 1$.
For $i = n$ and if $j = 2, 3, \ldots, k$,

$$\begin{aligned} f(u_{n,j}) &= f(u_{n-1,j}) + 2 \quad \text{if } f(u_{n-1,j}) \leq n-2 \\ &= 1 \quad \text{otherwise} \end{aligned}$$

Step 4: If n is even,

(i) Label the vertices of the first page $u_{1,1}, u_{1,2}, u_{1,3}, \ldots, u_{1,k}$ with the labels $2, 4, 6, \ldots, n$ cyclically.

(ii) Label the vertices of the i-th page, $i = 3, 5, \ldots, n-1$ by

$$\begin{aligned} f(u_{i,p}) &= f(u_{i-2,p}) + 2 \quad \text{if } f(u_{i-2,p}) \leq n-2 \\ &= 2 \qquad\qquad\qquad \text{otherwise} \end{aligned}$$

(iii) Label the vertices of the second page $u_{2,1}, u_{2,2}, u_{2,3}, \ldots, u_{2,k}$ with the labels $3, 5, 7, \ldots, n-1, 1$ cyclically.

(iv) Label the vertices of the i-th page, $i = 4, 6, \ldots, n-2$ by

$$\begin{aligned} f(u_{i,p}) &= f(u_{i-2,p}) + 2 \quad \text{if } f(u_{i-2,p}) \leq n-3 \\ &= 1 \qquad\qquad\qquad \text{otherwise} \end{aligned}$$

(v) Label the vertices of the n-th page by,
For $i = n$ and $j = 1$, $f(u_{n,j}) = f(u_{n-2,j}) + 2$.
For $i = n$ and if $j = 2, 3, \ldots, k$,

$$\begin{aligned} f(u_{n,j}) &= f(u_{n-2,j}) + 2 \quad \text{if } f(u_{n-2,j}) \leq n-3 \\ &= 1 \qquad\qquad\qquad \text{otherwise} \end{aligned}$$

□

Next, we consider the graph, a collection of n copies of C_4s, all of which have a common vertex, that is, nC_4 with a common vertex. Now, subdivide the edges of C_4s such that each cycle has the same number of vertices after subdivisions. So, the graph nC_4 with a common vertex after subdivisions is nC_α with the common vertex.

Theorem 3. *The λ-number of nC_α with a common vertex is $2n+1$, $n > 2$.*

Proof. Let G be the graph, nC_α with a common vertex. Let the common vertex be u and the vertices of the i-th cycle be $u, v_i, v_{i,1}, v_{i,2}, \ldots, v_{i,k}, w_i, u$, $i = 1, 2, \ldots, n$.
Define $f : V(G) \to \{0, 1, 2, \ldots, 2n+1\}$ as follows.

$$\begin{aligned} f(u) &= 0 \\ f(v_i) &= 2i, \qquad\qquad i = 1, 2, \ldots, n \\ f(w_i) &= f(v_i) + 1, \quad i = 1, 2, \ldots, n \end{aligned}$$

Consider the vertices $v_{i,j}$, $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, k$.
If $j \equiv p(\bmod 3)$,

$$
\begin{array}{llcl}
\text{For } i = 1, & f(v_{i,j}) & = & 5 \text{ if } p = 1 \\
 & & = & 1 \text{ if } p = 2 \\
 & & = & 7 \text{ if } p = 0 \\
\text{For } i = 2, & f(v_{i,j}) & = & 1 \text{ if } p = 1 \\
 & & = & 3 \text{ if } p = 2 \\
 & & = & 7 \text{ if } p = 0 \\
\text{For } i = 3, 4, \ldots, n & f(v_{i,j}) & = & 1 \text{ if } p = 1 \\
 & & = & 3 \text{ if } p = 2 \\
 & & = & 5 \text{ if } p = 0
\end{array}
$$

Now we prove that f is distance two labeling.

We note that $d(u, v_i) = d(u, w_i) = 1$, $i = 1, 2, \ldots, n$. Since $f(u) = 0$, $f(v_i) = 2i$ and $f(w_i) = 2i + 1$, $i = 1, 2, \ldots, n$, we have,

$$|f(u) - f(v_i)| = 2i, \text{ and } |f(u) - f(w_i)| = 2i + 1, \; i = 1, 2, \ldots, n.$$

For all other vertices which are at a distance 1, the absolute value of their label difference is greater than or equal to 2 is immediate, by construction of f.

For $i, j = 1, 2, \ldots, n$, we note that, $d(v_i, w_j) = 2$, and for $i \neq j$, $d(v_i, v_j) = 2$, $d(w_i, w_j) = 2$. Since all the labels of v_is and w_is are different, the absolute value of their label difference is greater than or equal to 1. For all other vertices which are at a distance 2, the absolute value of their label difference is greater than or equal to 1 is immediate, by the construction of f. Hence f is a $L(2,1)$-Labeling and $\lambda(G) \leq 2n + 1$. Since the maximum degree of G is $2n$, $\lambda(G) \geq 2n + 1$. Hence $\lambda(G) = 2n + 1$. □

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