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Uniform Stability and Boundedness of a Kind of Third Order Delay Differential Equations

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Abstract

By constructing a Lyapunov functional, we obtain some sufficient conditions which guarantee the stability and boundedness of solutions for some nonlinear differential equations of third order with delay. Our results improve and extend some well known results in the literature and one example is given for illustration of the subject.

Keywords: stability, Lyapunov functional, delay differential equations, third-order differential equations.

1 Introduction

We consider nonlinear third order delays differential equations of the form

$$[h(x(t))x'(t)]'' + a(t)\psi(x'(t))x''(t) + b(t)g(x'(t)) + c(t)f(x(t-r)) = 0, \quad (1)$$

and

$$\begin{aligned}&[h(x(t))x'(t)]'' + a(t)\psi(x'(t))x''(t) + b(t)g(x'(t)) + c(t)f(x(t-r)) \\ &= p(t, x(t), x(t-r), x'(t), x'(t-r), x''(t)), \quad (2)\end{aligned}$$

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where $r > 0$, and the functions $a(t), b(t), c(t), g(x), h(x), \psi(x'), f(x)$, and $p(t, x(t), x(t-r), x'(t), x'(t-r), x''(t))$ are continuous in their respective arguments. Besides, it is supposed that the derivatives $f'(x)$, $g'(y)$ are continuous for all x, y with $f(0) = g(0) = 0$. In addition, it is also assumed that the functions $f(x(t-r))$, $g(y(t))$ and $p(t, x, x', x(t-r), x'(t-r), x'')$ satisfy a Lipschitz condition in $x, x', x(t-r), x'(t-r)$ and x''; throughout the paper $x(t), y(t)$ and $z(t)$ are, respectively, abbreviated as x, y and z. Then the solution is unique (See [1] p.14).

In recent years, many books and papers dealt with the delay differential equation and obtained many good results, for example, [2–9], etc. In many references, the authors dealt with the problems by considering Lyapunov functions or functionals and obtained the criteria for the stability and boundedness (See [1–17]).

In 2009, [3] investigated the asymptotic behavior and boundedness of solutions of the following nonlinear differential equation, with a constant deviating argument r,

$$x'''(t) + a(t)x''(t) + b(t)g(x'(t)) + c(t)f(x(t-r)) = p(t).$$

More recently, Tunç [6], obtained sufficient conditions which ensure the boundedness of the delay differential equation of the form

$$\begin{aligned} &x'''(t) + a(t)\psi(x'(t))x''(t) + b(t)g(x'(t)) + c(t)f(x(t-r)) \\ &= p(t, x(t), x(t-r), x'(t), x'(t-r), x''(t)). \end{aligned} \tag{3}$$

The purpose of this paper is to extend the results verified by Tunç [6]. Whenever $h(x(t)) = 1$, equation (2) reduces to (3) and the hypotheses and conclusions of our theorem coincide with that discussed by Tunç [6]. We shall utilize Lyapunov functional and we give sufficient conditions for the uniform asymptotic stability and boundedness of solutions of delay differential equation (2) for the cases $p \equiv 0$ and $p \neq 0$.

2 Stability

First, we will give some basic definitions and important stability criteria for the general non-autonomous delay differential system. We consider

$$x' = f(t, x_t), \quad x_t(\theta) = x(t+\theta)\ , \quad -r \leq \theta \leq 0, \quad t \geq 0, \tag{4}$$

where $f : I \times C_H \to \mathbb{R}^n$ is a continuous mapping, $f(t,0) = 0$, $C_H := \{\phi \in (C[-r,0],\ \mathbb{R}^n) : \|\phi\| \leq H\}$, and for $H_1 < H$, there exists $L(H_1) > 0$, with $|f(t,\phi)| < L(H_1)$ when $\|\phi\| < H_1$.

Definition 2.1. *[10] An element $\psi \in C$ is in the $\omega-$limit set of ϕ, say $\Omega(\phi)$, if $x(t,0,\phi)$ is defined on $[0,+\infty)$ and there is a sequence $\{t_n\}, t_n \to \infty$, as $n \to \infty$, with $\|x_{t_n}(\phi) - \psi\| \to 0$ as $n \to \infty$ where $x_{t_n}(\phi) = x(t_n + \theta, 0, \phi)$ for $-r \leq \theta \leq 0$.*

Definition 2.2. *[10] A set $Q \subset C_H$ is an invariant set if for any $\phi \in Q$, the solution of (4), $x(t,0,\phi)$, is defined on $[0,\infty)$ and $x_t(\phi) \in Q$ for $t \in [0,\infty)$.*

Lemma 2.1. *[1] If $\phi \in C_H$is such that the solution $x_t(\phi)$ of (4) with $x_0(\phi) = \phi$ is defined on $[0,\infty)$ and $\|x_t(\phi)\| \leq H_1 < H$ for $t \in [0,\infty)$, then $\Omega(\phi)$ is a non-empty, compact, invariant set and*

$$dist(x_t(\phi), \Omega(\phi)) \to 0 \quad as \quad t \to \infty.$$

Lemma 2.2. *[1] Let $V(t,\phi) : I \times C_H \to \mathbb{R}$ be a continuous functional satisfying a local Lipschitz condition.$V(t,0) = 0$, and such that:*
(i) *$W_1(|\phi(0)|) \leq V(t,\phi) \leq W_2(\|\phi\|)$ where $W_1(r)$, $W_2(r)$ are wedges.*
(ii) *$V'_{(4)}(t,\phi) \leq 0$, for $\phi \in C_H$.*
Then the zero solution of (4) is uniformly stable.
If $Z = \{\phi \in C_H : V'_{(4)}(t,\phi) = 0\}$, then the zero solution of (4) is asymptotically stable, provided that the largest invariant set in Z is $Q = \{0\}$.

3 Assumptions and main results

Suppose that there are positive constants $a, b, c, d, d_0, d_1, h_0, h_1, \delta_0, \delta_1, \delta_2$ and $\beta, A, B, C, L, \sigma$. Our basic assumption is the following:

i) $0 < a \leq a(t) \leq A;\ 0 < b \leq b(t) \leq B;\ 0 < c \leq c(t) \leq C,$

ii) $c(t) \leq b(t),\ -L \leq b'(t) \leq c'(t) \leq 0$ for $t \in [0,\infty)$,

iii) $1 \leq \psi(y) \leq \beta;\ \ 0 < h_0 \leq h(x) \leq h_1,$

iv) $\dfrac{f(x)}{x} \geq \delta_0 > 0\ (x \neq 0)$, and $|f'(x)| \leq \delta_1$ for all x,

v) $d_1 \geq \dfrac{g(y)}{y} \geq d_0 > 0 \ (y \neq 0)$,

vi) $\dfrac{h_1\delta_1}{d_0} < d < a$,

vii) $\frac{1}{2}a'(t) \leq \delta_2 < \dfrac{b(dd_0 - \delta_1 h_1)}{d\beta}$,

viii) $\int_{-\infty}^{+\infty} |h'(u)|\, du < \infty$.

Theorem 3.1. *Under the conditions stated above, assume that the following is also satisfied*

$$r < \min\left\{\frac{2(a-d)}{h_1 C\delta_1}, \frac{2\sigma h_0^3}{C\delta_1 h_1^2(d + dh_0^2 + h_0)}\right\},$$

where

$$d\delta_2\beta + b(\delta_1 h_1 - dd_0) = -\sigma < 0.$$

Then every solution of (1) is uniformly asymptotically stable.

Proof. To verify Theorem 3.1, we write the Eq. (1) as the following equivalent system:

$$\begin{aligned} x' &= \frac{1}{h(x)}y \\ y' &= z \\ z' &= -\frac{a(t)}{h(x)}z\psi(\frac{y}{h(x)}) + \frac{a(t)h'(x)}{h^3(x)}y^2\psi(\frac{y}{h(x)}) - b(t)g(\frac{y}{h(x)}) \\ &\quad -c(t)f(x) + c(t)\int_{t-r}^{t}\frac{y(s)}{h(x(s))}f'(x(s))ds. \end{aligned} \tag{5}$$

For abbreviation, we write $\theta(t)$ instead of $\dfrac{h'(x(t))}{h^2(x(t))}x'(t)$.

Our main tool in the proof of the theorem just stated above is a Lyapunov function $U = U(t, x_t, y_t, z_t)$ defined by

$$U(t, x_t, y_t, z_t) = \exp\left(-\frac{1}{\mu}\int_0^t |\theta(s)|\, ds\right)V, \tag{6}$$

where

$$\begin{aligned} V &= dc(t)F(x) + c(t)f(x)y + b(t)h(x)G(\frac{y}{h(x)}) + \frac{1}{2}z^2 + \frac{d}{h(x)}yz \\ &+ da(t)\int_0^{\frac{y}{h(x)}} \psi(u)udu + \lambda \int_{-r}^0 \int_{t+s}^t y^2(\xi)d\xi ds, \end{aligned} \tag{7}$$

such that $F(x) = \int_0^x f(u)du$, and $G(y) = \int_0^y g(u)du$, μ and λ are positives constants which will be specified later in the proof. We can rewrite (7) as

$$\begin{aligned} V &= da(t)\int_0^{\frac{y}{h(x)}} [\psi(u) - \frac{d}{a(t)}]udu + \frac{1}{2}(z + \frac{d}{h(x)}y)^2 \\ &+ c(t)\left[dF(x) + \frac{b(t)h(x)}{c(t)}G(\frac{y}{h(x)}) + f(x)y\right] \\ &+ \lambda \int_{-r}^0 \int_{t+s}^t y^2(\xi)d\xi ds. \end{aligned}$$

On using the hypothesis (i)- (vi), we obtain

$$\begin{aligned} V &\geq da(1 - \frac{d}{a})\frac{y^2}{2h^2(x)} + \frac{1}{2}\left(z + \frac{d}{h(x)}y\right)^2 + \frac{c(t)h(x)}{2d_0}\left(\frac{d_0 y}{h(x)} + f(x)\right)^2 \\ &+ dc\int_0^x (1 - \frac{\delta_1 h_1}{dd_0})f(s)ds + \lambda \int_{-r}^0 \int_{t+s}^t y^2(\xi)d\xi ds. \end{aligned}$$

Since the integral $\lambda \int_{-r}^0 \int_{t+s}^t y^2(\xi)d\xi ds$ is positive,

$$\begin{aligned} V &\geq da(1 - \frac{d}{a})\frac{y^2}{2h^2(x)} + \frac{1}{2}\left(z + \frac{d}{h(x)}y\right)^2 + \frac{c(t)h(x)}{2d_0}\left(\frac{d_0 y}{h(x)} + f(x)\right)^2 \\ &+ \frac{\delta_3\delta_0}{2}x^2, \end{aligned}$$

where $\delta_3 = dc(1 - \dfrac{h_1\delta_1}{dd_0}) > dc(1 - \dfrac{d}{d}) = 0$.
Hence, there exists a positive constant k, small enough such that

$$V \geq k(x^2 + y^2 + z^2). \tag{8}$$

It is easy to check that by (iii) and (viii), we have

$$\begin{aligned} \int_0^t |\theta(s)|\, ds &= \int_{\alpha_1(t)}^{\alpha_2(t)} \frac{|h'(u)|}{h^2(u)}du \\ &\leq \frac{1}{h_0^2}\int_{-\infty}^{+\infty} |h'(u)|\, du \leq N < \infty, \end{aligned}$$

where $\alpha_1(t) = \min\{x(0), x(t)\}$, and $\alpha_2(t) = \max\{x(0), x(t)\}$.
Therefore we can find a continuous function $W_1(|\Phi(0)|)$ with

$$W_1(|\Phi(0)|) \geq 0 \quad \text{and} \quad W_1(|\Phi(0)|) \leq U(t, \Phi).$$

The existence of a continuous function $W_2(\|\phi\|)$ which satisfies the inequality $U(t, \phi) \leq W_2(\|\phi\|)$, is easily verified.

Now, along a trajectory of (5) we find

$$\begin{aligned}
\frac{d}{dt}V_{(5)} &= dc'(t)F(x) + c'(t)yf(x) + b'(t)h(x)G(\frac{y}{h(x)}) \\
&+ \frac{d}{h(x)}z^2 - a(t)\frac{z^2}{h(x)}\Psi(\frac{y}{h(x)}) - db(t)\frac{y}{h(x)}g(\frac{y}{h(x)}) \\
&+ da'(t)\int_0^{\frac{y}{h(x)}} \Psi(u)udu + \frac{c(t)f'(x)}{h(x)}y^2 + \lambda ry^2 \\
&+ \theta(t)\left[b(t)h^2(x)G(\frac{y}{h(x)}) - b(t)yh(x)g(\frac{y}{h(x)}) + (a(t)\Psi(\frac{y}{h(x)}) - d)zy\right] \\
&+ c(t)(\frac{dy}{h(x)} + z)\int_{t-r}^{t} y(s)\frac{f'(x(s))}{h(x(s))}ds - \lambda\int_{t-r}^{t} y^2(\xi)d\xi.
\end{aligned}$$

Since $|f'(x)| \leq \delta_1$, we obtain the following inequalities

$$\frac{dc(t)}{h(x)}y\int_{t-r}^{t}\frac{y(s)}{h(x(s))}f'(x(s))ds \leq \frac{C\delta_1 dr}{2h_0}y^2 + \frac{Cd\delta_1}{2h_0^3}\int_{t-r}^{t} y^2(\xi)d\xi,$$

and

$$c(t)z\int_{t-r}^{t}\frac{y(s)}{h(x(s))}f'(x(s))ds \leq \frac{C\delta_1 r}{2}z^2 + \frac{C\delta_1}{2h_0^2}\int_{t-r}^{t} y^2(\xi)d\xi.$$

By using the conditions (i)-(vii) and the last inequalities, we have

$$\begin{aligned}
\frac{d}{dt}V_{(5)} &\leq dc'(t)F(x) + c'(t)yf(x) + b'(t)h(x)G(\frac{y}{h(x)}) \\
&- \left[\frac{a-d}{h_1} - \frac{C\delta_1 r}{2}\right]z^2 + \left[\frac{d\delta_2\beta + b(\delta_1 h_1 - dd_0)}{h_1^2}\right]y^2 \\
&+ |\theta(t)|\left[\frac{3}{2}d_1By^2 + (A\beta - d)|zy|\right] + (\lambda + \frac{dC\delta_1}{2h_0})ry^2 \\
&+ \left[\frac{C\delta_1}{2h_0^2}(1 + \frac{d}{h_0}) - \lambda\right]\int_{t-r}^{t} y^2(\xi)d\xi.
\end{aligned}$$

We claim that

$$Q(t,x,y) = dc'(t)F(x) + c'(t)yf(x) + b'(t)h(x)G(\frac{y}{h(x)}) \leq 0,$$

for all x, y and $t \geq 0$. There are two cases $c'(t) = 0$ or $c'(t) < 0$.
Case 1: $c'(t) = 0$, then $Q(t,x,y) = \dfrac{d_0 b'(t)}{2h(x)} y^2 \leq 0$.
Case 2: $c'(t) < 0$, observe that (v) implies that

$$G(y) \geq \frac{1}{2} d_0 y^2,$$

hence

$$\begin{aligned} Q(t,x,y) &\leq dc'(t) \left[F(x) + \frac{1}{d} yf(x) + \frac{d_0 b'(t)}{2dh(x)c'(t)} y^2 \right] \\ &\leq dc'(t) \left[F(x) + \frac{d_0 b'(t)}{2dh(x)c'(t)} \left\{ y + \frac{c'(t)h(x)f(x)}{d_0 b'(t)} \right\}^2 - \frac{c'(t)h(x)f^2(x)}{2dd_0 b'(t)} \right] \end{aligned}$$

It is required that $\dfrac{c'(t)}{b'(t)} \leq 1$ by (ii), then

$$\begin{aligned} Q(t,x,y) &\leq dc'(t) \int_0^x (1 - \frac{h_1 \delta_1}{dd_0}) f(u) du \\ &\leq c'(t) \frac{\delta_3}{c} F(x) \leq 0. \end{aligned}$$

In both cases, we have $Q(t,x,y) \leq 0$ for all $t \geq 0$, x and y. Using $2|uv| \leq (u^2 + v^2)$, we obtain

$$\begin{aligned} |\theta(t)| \left[\frac{3d_1}{2} By^2 + (A\beta - d)|zy| \right] &\leq \frac{1}{2} |\theta(t)| \left[3d_1 By^2 + (A\beta - d)(y^2 + z^2) \right] \\ &\leq k_1 |\theta(t)| (y^2 + z^2), \end{aligned}$$

where $k_1 = \dfrac{1}{2}(A\beta - d + 3d_1 B)$. Consequently, we get

$$\begin{aligned} \frac{d}{dt} V_{(5)} \leq & - \left[\frac{\sigma}{h_1^2} - (\lambda + \frac{dC\delta_1}{2h_0}) r \right] y^2 - \left[\frac{a-d}{h_1} - \frac{C\delta_1 r}{2} \right] z^2 \\ & + \left[\frac{C\delta_1}{2h_0^2} (1 + \frac{d}{h_0}) - \lambda \right] \int_{t-r}^{t} y^2(\xi) d\xi \\ & + k_1 |\theta(t)| (y^2 + z^2). \end{aligned}$$

Replacing λ by $\frac{C\delta_1}{2h_0^2}(1+\frac{d}{h_0})$, the last inequality becomes

$$\begin{aligned}\frac{d}{dt}V_{(5)} &\leq -\left[\frac{\sigma}{h_1^2}-\frac{C\delta_1}{2h_0}(d+\frac{1}{h_0}+\frac{d}{h_0^2})r\right]y^2\\ &\quad -\left[\frac{a-d}{h_1}-\frac{C\delta_1 r}{2}\right]z^2\\ &\quad +k_1\left|\theta(t)\right|(y^2+z^2).\end{aligned}$$

Combining (6) with (8) and let $\mu=\frac{k}{k_1}$, we obtain:

$$\begin{aligned}\frac{d}{dt}U_{(5)} &= \exp\left(-\frac{k_1}{k}\int_0^t\left|\theta(s)\right|ds\right)\left(\frac{d}{dt}V_{(5)}-\frac{k_1|\theta(t)|}{k}V\right)\\ &\leq \exp\left(-\frac{k_1}{k}\int_0^t\left|\theta(s)\right|ds\right)(-k_2y^2-k_3z^2). \qquad (9)\end{aligned}$$

where $k_2=\frac{\sigma}{h_1^2}-\frac{C\delta_1 r}{2h_0}(d+\frac{1}{h_0}+\frac{d}{h_0^2})$ and $k_3=\frac{a-d}{h_1}-\frac{C\delta_1 r}{2}$.
Choosing

$$r<\min\left\{\frac{2(a-d)}{h_1C\delta_1},\frac{2\sigma h_0^3}{C\delta_1h_1^2(d+dh_0^2+h_0)}\right\},$$

we have

$$\frac{d}{dt}U(t,x_t,y_t,z_t)\leq-\alpha\exp(-\frac{k_1N}{k})(y^2+z^2),\quad \text{for some}\quad \alpha>0.$$

Thus, under the above discussion, we conclude that the trivial solution of equation (1) is uniformly asymptotically stable. This fact completes the proof. □

Example. *We consider the following third order non-autonomous delay differential equation*

$$\begin{aligned}&\left[(\frac{\sin x}{1+x^2}+2)x'\right]''+(\frac{1}{4}\sin t+10)(\arctan x'+\pi)x'' \qquad (10)\\ &+(\frac{1}{1+t}+1)(2x'+\frac{x'}{1+x'^2})x'\\ &+\frac{1}{4}(\frac{1}{1+t}+1)(x(t-r)+\frac{x(t-r)}{1+x^2(t-r)})=0.\end{aligned}$$

Now, it is easy to see that

$$\frac{39}{4} = a \le a(t) = \frac{1}{4}\sin t + 10 \le \frac{41}{4}, \quad a'(t) = \frac{1}{4}\cos t \le \frac{1}{4} \quad \textit{for all } t \ge 0,$$

$$1 = b \le b(t) = \frac{1}{1+t} + 1 \le 2, \quad \frac{1}{4} \le c(t) = \frac{1}{4(1+t)} + \frac{1}{4} \le \frac{1}{2},$$

$$1 \le h(x) = \frac{\sin x}{1+x^2} + 2 \le 3 = h_1,$$

$$1 \le \frac{f(x)}{x} = 1 + \frac{1}{1+x^2} \ \textit{with } x \ne 0, \ \textit{and } |f'(x)| \le 2 = \delta_1,$$

$$1 \le \psi(y) = \arctan y + \pi \le 5 = \beta,$$

$$2 = d_0 \le \frac{g(y)}{y} = 2 + \frac{1}{1+y^2} \le 3 \ \textit{with } y \ne 0,$$

$$\frac{h_1\delta_1}{d_0} = 3 < d < \frac{39}{4} = a,$$

$$\frac{1}{8}\cos t < \frac{b(dd_0 - h_1\delta_1)}{d^2\beta} < \frac{3}{10}.$$

A sample calculation shows

$$\begin{aligned} \int_{-\infty}^{+\infty} |h'(u)|\, du &\le \int_{-\infty}^{+\infty} \left[\left| \frac{\cos u}{1+u^2} \right| + \left| \frac{2u \sin u}{(1+u^2)^2} \right| \right] du \\ &\le \pi + 2. \end{aligned}$$

All the assumptions (i) through (viii) are satisfied, we can conclude using Theorem 3.1 that every solution of (10) is uniformly asymptotically stable.

In the case $p \ne 0$ we establish the following result:

Theorem 3.2. *In addition to the assumptions of Theorem 3.1, If we assume that*

$$|p(t, x, x(t-r), y, y(t-r), z)| \le q(t),$$

where $q \in L^1(0,\infty)$, $L^1(0,\infty)$ is the space of Lebesgue integrable functions. Then all solutions of the perturbed equation (2) are bounded.

Proof. The remaining of this proof follows the strategy indicated in the proof of Theorem in [6] and hence it omitted. □

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