CompAMa

www.compama.co.usb.ve

Bull. Comput. Appl. Math., Vol.1, No.2, 2013

ISSN 2244-8659

Numerical solution for a family of delay functional differential equations using step by step Tau approximations

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CompAMa Vol.1, No.2, pp.81-91, 2013 - Accepted December 30, 2013

Abstract

We use the segmented formulation of the Tau method to approximate the solutions of a family of linear and nonlinear neutral delay differential equations

$$\begin{aligned} a_1(t)y'(t) &= y(t)[a_2(t)y(t-\tau)+a_3(t)y'(t-\tau)+a_4(t)] \\ &\quad + a_5(t)y(t-\tau)+a_6(t)y'(t-\tau)+a_7(t), \quad t\geq 0 \\ y(t) &= \Psi(t), \quad t\leq 0 \end{aligned}$$

which represents, for particular values of $a_i(t)$, $i=1,7$, and τ, functional differential equations that arise in a natural way in different areas of applied mathematics. This paper means to highlight the fact that the step by step Tau method is a natural and promising strategy in the numerical solution of functional differential equations.

Keywords: Functional differential equations; Step by step Tau method; Neutral delay differential equations.

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†This author was partially supported by the Centro de Estadística y Software Matemático (CESMa) and the Decanato de Investigación y Desarrollo (DID) at USB.

1 Preliminaries

In this paper the segmented Lanczos-Tau method (i.e. step by step Tau method [1]) is used to find numerical solutions of the nonhomogeneous functional differential problem of neutral type:

$$\begin{aligned} a_1(t)y'(t) &= y(t)[a_2(t)y(t-\tau) + a_3(t)y'(t-\tau) + a_4(t)] \\ &\quad + a_5(t)y(t-\tau) + a_6(t)y'(t-\tau) + a_7(t), \quad t \geq 0 \\ y(t) &= \Psi(t), \quad t \leq 0, \end{aligned} \tag{1}$$

where $\tau > 0$, and the $a_i(t)$'s and $\Psi(t)$ are polynomial functions in $[0, \infty)$ and $[-\tau, 0]$, respectively (on the contrary, we use piecewise polynomial approximations of the given functions according to their smoothness). Eq. (1) has as particular cases: the neutral ($a_3(t) \neq 0$ or $a_6(t) \neq 0$), delay ($a_3(t) \equiv a_6(t) \equiv 0$), linear ($a_2(t) \equiv a_3(t) \equiv 0$) and nonlinear (at least one of the $a_i(t)$'s, $i = 2, 3$, is nonzero) functional differential equations.

In two recent papers, [2] and [3], linear and nonlinear cases were studied separately, and in [4] and [5], particular cases were also considered using the segmented Tau method. To details about some linear and nonlinear particular cases see [2]-[3] and the references therein.

Since for (1) no closed form of analytical solution is available (except for linear case [6]), we obtain polynomial approximations to its solutions by applying a spectral technique, the *step by step Tau method* (or SST method to abbreviate) with the adopted approach in the papers [2]-[5].

The *Tau method*, first introduced by Lanczos [7], is an important example of how to get approximations of functions defined by a differential equation. It is an important feature of the Tau method that no trial solutions, approximate quadratures or large matrix inversions are required. In the formulation of a *step by step* Tau version it is allowed to construct *piecewise polynomial approximations* of a given function which can be used to start a refining process (see [1] for details).

Next we shall briefly sketch the *Ortiz's recursive formulation of the Tau method* and its segmented version (i.e. the SST method).

The notation used here is, with slight modifications, that of [8]. Let P_j be the class of polynomials of degree less than or equal to j. Let us consider the equation defined by the differential operator L:

$$Ly(x) \equiv \sum_{i=0}^{\nu} p_i(x) y^{(i)}(x) = f(x), \tag{2}$$

where $p_i(x)$ and $f(x)$ are polynomials of finite degree (on the contrary, they are polynomial approximations of given functions), and $y^{(i)}$ represents the ith derivative of $y(x)$. We assume also that the solution $y(x)$ of (2) satisfies *supplementary conditions*

$$(f_k, y) = s_k, \quad k = 1, \ldots, \nu, \quad x \in [a,b], \tag{3}$$

where $[a,b]$ is a compact interval, f_k are linear functionals acting on $y(x)$.

In order to express the Tau method approximations as a weighted arithmetic mean of successive partial sums of the power series solutions and obtaining approximations of high precision, Lanczos proposed ([9]) the concept of the *canonical polynomials* $Q_n(x)$, $n \in I\!N_0 \equiv I\!N \cup \{0\}$, associated with a linear differential operator. Afterwards, Ortiz introduced the more workable definition of $Q_n(x)$ [8], $LQ_n(x) = x^n + R_n$, where $R_n(x)$ is a polynomial generated by $\{x^i\}$, $i \in S$ (set of indices for which canonical polynomials remain undefined), and is called the *residual polynomial* of $Q_n(x)$. *Note*: $\mathbf{R}_S \equiv \mathrm{span}_{i\in S}\{x^i\}$.

Another important concept is that of *generating polynomials* [8], which are obtained from applying L to x^n, where $n \in I\!N_0$. From them we can find a recursive relation for the canonical polynomials. Each differential operator L (that belongs to the class characterized by (2)) is uniquely associated with a sequence $\{Q_n\}$, $n \in I\!N_0 - S$, of canonical polynomials. For further details about how to generate this sequence and precise definitions of Ortiz' algebraic theory of the Tau method we refer to [8].

Let $\mathbf{v} = \{v_i(x)\} = V\mathbf{x}$ be a polynomial basis defined by a lower triangular matrix $V = (v_{ij})$, with $i,\ j \in I\!N$, acting on $\mathbf{x} = (1, x, x^2, \ldots)^t$. Let $\hat{Q} = \{\hat{Q}_n(x)\}$, $n \in I\!N - S$ such that $L\hat{Q}_n(x) = v_n(x) + \hat{R}_n(x)$, and $\hat{Q}_n(x) = \sum_{j=0}^{n} v_n Q_j(x)$, where $j \in S$. We consider the perturbed equation

$$Ly_n(x) = f(x) + H_n(x), \tag{4}$$

where $H_n(x) = \sum_{i=0}^{m} \tau_i^{(n)} v_{n-i}(x) \in P_n$, which is called the *perturbation term*, is expressed in terms of the basis $\mathbf{v}$. The $\tau_i^{(n)}$'s are parameters that we wish to find. Further we assume that $f(x) = \sum_{i=0}^{r} \alpha_i v_i(x)$, where $r \leq n$. Hence,

$$y_n(x) = \sum_{i=0}^{m} \tau_i^{(n)} \hat{Q}_{n-i}(x) + \sum_{i=0}^{r} \alpha_i \hat{Q}_i(x), \tag{5}$$

where $i \notin S$. Equation (5) is called the *Tau approximant* of order n of $y(x)$, and satisfies exactly (4) and (3). ν Tau parameters are chosen such that y_n

satisfies exactly conditions (3). Additional Tau parameters (say s) are chosen such that the residuals of $Ly_n(x)$ match the components of $f(x) + H_n(x)$ belonging to $\mathbf{R}_S$. If there exist t exact polynomial solutions of (2), then $s + \nu - t - 1 = m$ (see [8] for details).

We observe that the generation of the canonical polynomials depend neither on the conditions (3) nor on the interval in which the solution is required. These features allow us to introduce the concept of Ortiz's segmented approximations.

Let Π be a partition of the interval $[a, b]$ into subintervals $I_j = [x_{j-1}, x_j]$, $j = 1, \ldots, E$, where $x_0 = a < x_1 < x_2 < \ldots < x_E = b$. We consider, for $j = 1, \ldots, E$, the Tau approximants $y_{n_j}(x)$, with $x \in I_j$, which define a *piecewise Tau approximant* $y_n(x)$ of order n of the solution $y(x)$ of problem (2)-(3), if each of the y_{n_j}'s satisfies (2) with a polynomial perturbation term $H_n^j(x)$, with $x \in I_j$, and, for $k = 1, \ldots, \nu$, $(f_k, y_{n_r}) = s_k$ $(r = 1, E)$, and, for $j = 2, \ldots, E$, $y_{n_{j-1}}^{(i)}(x_{j-1}) = y_{n_j}^{(i)}(x_{j-1})$, $i = 0, \ldots, \nu - 1$. We point out that the construction of a piecewise Tau approximation $y_n(x)$ of the solution $y(x)$ of problem (2)-(3) depends only on one matrix V and one canonical sequence Q.

There is a range of possibilities in the choice of a basis $\mathbf{v}$. Lanczos in [9] and [10] suggested the choice of Chebyshev and Legendre polynomials to obtain a better distribution of errors over the interval in which the original problem is defined and the approximate solution is required. In particular, Lanczos' remark concerning the approximations obtained using a Legendre polynomial perturbation term, allows us to obtain accurate estimations at the end point of that interval [10], which is the key fact to construct the step by step formulation of the Tau method in which the error is minimized at the matching point of successive steps [1].

This paper is organized as follows. In Section 2, we find the piecewise polynomial approximation of the involved neutral problem using the Lanczos-Tau method, and we solve the outlined linear and nonlinear neutral differential problems using the recursive formulation of the Tau method. Finally, in Section 3 we present some concluding remarks.

2 Solving the differential equation using the SST method

We will use the following notation. The symbol I will denote the unit interval $[0,1]$ and, for each integer k, $k \geq -1$, the interval $[k\tau, (k+1)\tau]$ will be denoted by I_k. We apply the Tau method to each finite interval I_k of the domain and we solve the given problem in each interval separately.

Starting with the continuous function Ψ given on $[-\tau, 0]$, which will be a polynomial function, the method generates a polynomial approximation on the next interval $[0, \tau]$. In the same way it proceeds into subsequent intervals, each one with the same length τ. The initial condition is satisfied by the approximate solution in the first interval, and the initial condition in each subsequent interval is taken to be the endpoint value of the previous solution. Thus, Equation (1) is defined over $[0, \infty)$ in a piecewise manner. Here, we notice that our segmented strategy will force to shift the intervals I_ks to the unit interval I (i.e. starting with the shift of the function Ψ given in $[-\tau, 0]$ to I, the method generates a polynomial defined in I that approximates the solution of (1) in I_0; from this last polynomial a polynomial defined in I is generated that approximates the solution of (1) in I_1; we proceed in similar way into subsequent intervals I_k, $k \geq 2$).

It is clear that if $t \in [-\tau, \infty)$, then $t \in I_k$ for some $k \geq -1$. Hence, $t = \tau(x+k)$, with $x \in I$. Then, we can define, for all $x \in I$,

$$\begin{cases} y_k(x) &= y(\tau(x+k)) = y(t), \quad k \geq 0, \\ y_{-1}(x) &= \Psi(\tau(x-1)), \\ A_{ik}(x) &= a_{ik}(\tau(x+k)), \quad i = 1,7, \quad k \geq 0, \end{cases} \tag{6}$$

where, for each i=1,7, and $k \geq 0$, a_{ik} represents the restriction of $a_i(t)$ to the interval I_k. Thus, Eq. (6) leads to the sequence of differential equations

$$\begin{aligned} A_{1k}(x)y'_k(x) &= [\tau A_{2k}(x)y_{k-1}(x) + A_{3k}(x)y'_{k-1}(x) + \tau A_{4k}(x)]y_k(x) \\ &\quad + \tau A_{5k}(x)y_{k-1}(x) + A_{6k}(x)y'_{k-1}(x) + \tau A_{7k}(x), x \in I,\ k \geq 0 \\ y_{-1}(x) &= \Psi(\tau(x-1)),\ \ x \in I, \end{aligned} \tag{7}$$

and to match the solutions, at the endpoints of the intervals, we impose the conditions $y_k(0) = y_{k-1}(1)$, $k \geq 0$.

Let n be a fixed integer greater or equal than $\max\{r, r_{lk}\}$, where $r = \text{degree}(y_{-1}(x))$ and $r_{lk} = \max_{1\leq i\leq 7} \text{degree}(A_{ik}(x))$, for each $k \geq 0$ (n is the desired degree for the approximating polynomials that we seek). From Equation (4), for each $k \geq 0$, we will consider the following perturbed form of Eq. (7),

$$\begin{aligned} A_{1k}(x)Y_k'(x) - [\tau A_{2k}(x)Y_{k-1}(x) + A_{3k}(x)Y_{k-1}'(x) + \tau A_{4k}(x)]Y_k(x) \\ = \tau A_{5k}(x)Y_{k-1}(x) + A_{6k}(x)Y_{k-1}'(x) + \tau A_{7k}(x) + H_n^{(k)}(x), \\ x \in I, \ k \geq 0, \end{aligned} \tag{8}$$

where $Y_{-1}(x) = \Psi(\tau(x-1))$ $(x \in I)$, and $Y_{k-1}(x)$, for each $k \geq 1$, is the polynomial solution from the previous interval. The perturbation term $H_n^{(k)}(x)$ is defined in terms of the shifted Legendre polynomial in I, $P_n^*(x)$, and the τ-parameters (see Section 1). They are chosen according to the characteristics of each particular case of (1) (e.g. in [2] and [3] different perturbation terms were considered). The choice of Legendre over Chebyshev polynomials is motivated by their superior endpoint accuracy (see comments at the end of Section 1, and [10]).

The polynomials we seek are assumed to be of the form,

$$Y_k(x) = \sum_{j=0}^{n} a_j^{(k)} x^j, \quad k \geq -1,$$

with the conditions

$$Y_k(0) = Y_{k-1}(1), \quad k \geq 0. \tag{9}$$

Next, for each $k \geq 0$, we define the linear differential operator $L^{(k)}$ to be

$$L^{(k)}(\cdot) = A_{1k}(x)\frac{d}{dx}(\cdot) - [\tau A_{2k}(x)Y_{k-1}(x) + A_{3k}(x)Y_{k-1}'(x) + \tau A_{4k}(x)](\cdot) \tag{10}$$

Thus, for each $k \geq 0$, the perturbed problem (8)-(9) becomes

$$\begin{aligned} L^{(k)}Y_k(x) &= \tau A_{5k}(x)Y_{k-1}(x) + A_{6k}(x)Y_{k-1}'(x) + \tau A_{7k}(x) + H_n^{(k)}(x) \\ Y_k(0) &= Y_{k-1}(1) \end{aligned}$$

From now on the particular choice of perturbation term and coefficients $a_i(t)$'s in (1) will establish the corresponding sequence of canonical polynomials (see Section 1). So, in each particular case, it will allow us to apply the Tau method efficiently.

2.1 Solving linear neutral differential equations

In linear case $a_2(t) \equiv a_3(t) \equiv 0$. We will simplify our exposition doing $a_1(t) \equiv 1$, and $a_4(t)$, $a_5(t)$ and $a_6(t)$ independent of time t (i.e. we will consider a_4, a_5 and a_6 constant parameters in (1)). So that, Eq. (1) is transformed in

$$\begin{aligned} y'(t) &= a_4 y(t) + a_5 y(t-\tau) + a_6 y'(t-\tau) + a_7(t), \quad t \geq 0 \\ y(t) &= \Psi(t), \quad t \leq 0, \end{aligned} \tag{11}$$

Thus, following [2], we will suppose that $y_{-1}(x) = \sum_{j=0}^{r} a_j^{(-1)} x^j$ and $A_{7k}(x) = \sum_{j=0}^{r_k} \gamma_j^{(k)} x^j, \quad k \geq 0$, where r and r_k are nonnegative integers, and we will choose the perturbation term $H_n^{(k)}(x)$ such that it is defined in terms of a shifted Legendre polynomial as $H_n^{(k)}(x) = \tau_k \sum_{j=0}^{n} C_j x^j$, where the C_j's are the coefficients of the shifted Legendre polynomial $P_n^*(x)$ of degree n in I.

In $a_4 \neq 0$ case, let us define the linear operator L to be $L(\cdot) = \frac{d}{dx}(\cdot) - a_4\tau(\cdot)$. So, the perturbed equation (8) becomes

$$\begin{aligned} LY_k(x) &= a_5 \tau Y_{k-1}(x) + a_6 Y'_{k-1}(x) + \sum_{j=0}^{r_k} \tau \gamma_j^{(k)} x^j + \tau_k \sum_{j=0}^{n} C_j x^j \\ &= \sum_{j=0}^{n} (\beta_j^{(k-1)} + \tau_k C_j) x^j, \end{aligned} \tag{12}$$

where coefficients $\beta_j^{(k-1)}$ $(0 \leq j \leq n)$ are defined in terms of a_5, a_6, $a_j^{(k-1)}$ and $\gamma_j^{(k)}$ parameters (see [2] for details).

On the other hand, from the generating polynomials (see Section 1) it is easy to obtain the canonical polynomials. These are defined by

$$Q_m(x) = -\frac{m!}{a\tau} \sum_{i=0}^{m} \frac{1}{(a\tau)^{m-i} i!} x^i, \quad m \geq 0. \tag{13}$$

Note that there are no undefined canonical polynomials. Therefore, the exact polynomial solution of the perturbed equation (12) is set to be

$$Y_k(x) = \sum_{j=0}^{n} (\beta_j^{(k-1)} + \tau_k C_j) Q_j(x).$$

Now, if we substitute (13) in the above expression we have

$$Y_k(x) = -\frac{1}{a\tau}\sum_{j=0}^{n}\left(\sum_{i=0}^{j}\frac{(\beta_j^{(k-1)}+\tau_k C_j)j!}{(a\tau)^{j-i}i!}x^i\right),$$

and the exact polynomial solution of the perturbed equation (12) can be reduced to

$$Y_k(x) = -\frac{1}{a\tau}\sum_{j=0}^{n}\left(\sum_{i=j}^{n}\frac{(\beta_i^{(k-1)}+\tau_k C_i)i!}{(a\tau)^{i-j}j!}\right)x^j. \quad (14)$$

Now, for each $k \geq 0$, the parameter τ_k was introduced in order to account for the initial condition imposed at $x = 0$. Therefore, by setting $x = 0$ in (14) we solve for each k, the equation $Y_k(0) = Y_{k-1}(1)$, to obtain τ_k.

Finally, from (14) we have the desired polynomial approximations to the solution of the functional differential equation (11):

$$y(t) \approx Y_k(x) = Y_k\Big(\frac{t}{\tau} - k\Big), \quad t \in I_k, \quad k \geq 0.$$

In the case $a_4 = 0$ we do not need to apply the Tau method directly (see [2] for details).

2.2 Solving nonlinear neutral equations

In nonlinear case at least one of $a_2(t)$ and $a_3(t)$ is nonzero. We will simplify our exposition doing $a_1(t) \equiv 1$, $a_2(t)$, $a_3(t)$ and $a_4(t)$ independent of time t (i.e. we will consider a_2, a_3 and a_4 constant parameters in (1)), and $a_5(t) \equiv a_6(t) \equiv a_7(t) \equiv 0$. So that, Eq. (1) is transformed in the nonlinear neutral functional differential problem

$$\begin{aligned} y'(t) &= y(t)[a_2 y(t-\tau) + a_3 y'(t-\tau) + a_4], \quad t \geq 0 \\ y(t) &= \Psi(t), \quad t \leq 0, \end{aligned} \quad (15)$$

From (10), for each $k \geq 0$, we define the linear differential operator

$$L^{(k)}(\cdot) = \frac{d}{dx}(\cdot) - [\tau a_2 Y_{k-1}(x) + a_3 Y'_{k-1}(x) + \tau a_4](\cdot)$$

Following [3], for each $k \geq 0$, let us consider the perturbed form of Eq. (7)

$$L^{(k)}Y_k(x) = H_n^{(k)}(x) = \left(\sum_{i=0}^{r_k} \tau_i^{(k)} x^i\right) P_n^*(x) = \left(\sum_{i=0}^{r_k} \tau_i^{(k)} x^i\right)\left(\sum_{j=0}^{n} C_j x^j\right)$$
$$= \sum_{i=0}^{r_k}\left(\sum_{j=0}^{i} \tau_j^{(k)} C_{i-j}\right) x^i + \sum_{i=1}^{n-r_k}\left(\sum_{j=0}^{r_k} \tau_j^{(k)} C_{r_k+i-j}\right) x^{r_k+i}$$
$$+ \sum_{i=1}^{r_k}\left(\sum_{j=n-r_k}^{n-i} \tau_{n-j}^{(k)} C_{i+j}\right) x^{n+i}, \qquad (16)$$

with $x \in I$ and r_k $(0 \leq r_k \leq n)$ denoting the degree of $Y_{k-1}(x)$.

Next, from the generating polynomials we obtain the set of canonical polynomials $\{Q_m^{(k)}(x)\}$ associated with $L^{(k)}$. We distinguish two cases: $r_k = 0$ and $r_k \neq 0$. For the first one we have (whenever $a_0^{(k-1)} = Y_{k-1}(x) \neq -a_4 a_2^{-1}$)

$$Q_m^{(k)}(x) = -\frac{m!}{\tau(a_4 + a_2 a_0^{(k-1)})} \sum_{\nu=0}^{m} \frac{1}{\nu![\tau(a_4 + a_2 a_0^{(k-1)})]^{m-\nu}} x^\nu, \quad m \geq 0.$$

For $r_k \neq 0$ case we can obtain the following recursive definition ([3])

$$Q_{r_k+m}^{(k)}(x) = -\frac{1}{\tau a_2 a_0^{(k-1)}} \Bigg(x^m - m Q_{m-1}^{(k)}(x)$$
$$+ (\tau a_4 + \tau a_2 a_0^{(k-1)} + a_3 a_1^{(k-1)}) Q_m^{(k)}(x)$$
$$+ \sum_{\nu=1}^{r_k-1} [\tau a_2 a_\nu^{(k-1)} + a_3(\nu+1) a_{\nu+1}^{(k-1)}] Q_{\nu+m}^{(k)}(x) \Bigg), \quad m \geq 0. \qquad (17)$$

Note that the first r_k canonical polynomials $Q_i^{(k)}(x)$ $(0 \leq i \leq r_k - 1)$ remain undefined. From these two last expressions we obtain directly the exact polynomial solution of the perturbed equation (16). For each $k \geq 0$, it is set to be

$$Y_k(x) = \sum_{i=0}^{r_k}\left(\sum_{j=0}^{i} \tau_j^{(k)} C_{i-j}\right) Q_i^{(k)}(x) + \sum_{i=1}^{n-r_k}\left(\sum_{j=0}^{r_k} \tau_j^{(k)} C_{r_k+i-j}\right) Q_{r_k+i}^{(k)}(x)$$
$$+ \sum_{i=1}^{r_k}\left(\sum_{j=n-r_k}^{n-i} \tau_{n-j}^{(k)} C_{i+j}\right) Q_{n+i}^{(k)}(x). \qquad (18)$$

Here we can either obtain only $\tau_0^{(k)}$ from (9) ($r_k = 0$ case) or obtain the r_k+1 unknowns $\tau_j^{(k)}$ to account for the initial condition (9) in addition to the r_k undefined canonical polynomials ($r_k \neq 0$ case) . See [3] for details.

Finally, we obtain the polynomial approximations to the solution of the nonlinear neutral differential equation (15):

$$y(t) \approx Y_k(x) = Y_k\Big(\frac{t}{\tau} - k\Big), \quad t \in I_k, \quad k \geq 0.$$

3 Concluding remarks

In papers [2]-[5] intensive numerical experimentation have been carried out, and from them significant improvements have been obtained compared with the numerical results reported by other authors elsewhere, and using different approaches.

This paper, together with papers [2]-[5], demonstrate that the step by step Tau method is a natural and promising strategy for the numerical solution of functional differential equations.

The method proposed by us can be extended to more general problems (e.g. nonautonomous case and nonlinear systems). Recently, this approach was also applied to the case of a mixed-type functional differential equation problem with delayed and advance arguments [11].

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