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# The exponential distribution as the sum of discontinuous distributions

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**Abstract**

We show that for any natural number $n$, an exponential distribution can be written as the sum of $n$ discontinuous variables and another exponential distribution, all of them independent.

**Keywords:** infinitely decomposable, infinitely divisible

## 1 Introduction

The exponential distribution, whose density function is given by $f(x) = \alpha e^{-\alpha x}$ for $x \geq 0, \alpha > 0$, is one of the most studied in Probability theory and its applications. It models waiting times in a variety of scenarios, such as the Poisson process and queuing theory, specifically in the M/M/1 queue, where the times between arrivals and the times of service are all exponential. This latter scenario, described in Allen (1978), is the inspiration of this note.

It is well known that if $X$ is exponentially distributed with parameter $\alpha$ (and will say from now on that $X$ is $\exp(\alpha)$), it is is infinitely divisible, that is, for every natural number $n$ one can write

$$X = X_1 + \cdots + X_n, \tag{1}$$

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where the $X_i$s are independent identically distributed Gamma random variables with parameters $\alpha$ and $\frac{1}{n}$. For details about infinite divisibility and all other probabilistic concepts in this note, the reader is referred to Lamperti (1966) [2]. A weaker concept than infinite divisibility is that of infinite decomposability, where in equation (1) it is only required that the $X_i$s be independent but not necessarily identically distributed. We will show that every exponential variable is infinitely decomposable with the summands in (1) being all discontinuous random variables except for one. We believe that this result is not known and is certainly surprising.

## 2 The results

Given any $\mu > 0$ and $0 < \rho < 1$ we define the $\exp(\mu, \rho)$ random variable $Z$ as the one that has the following distribution: $P(Z = 0) = 1 - \rho$ and $P(Z > x) = \rho e^{-\mu(1-\rho)x}$ for all $x \geq 0$. In other words, the distribution of $Z$ has a jump at 0 with value $1 - \rho$, and other than that it looks like an exponential distribution. An $\exp(\mu, \rho)$ random variable $Z$ has a memoryless-like property:

$$P(Z > x + y | Z > x) = \frac{1}{\rho} P(Z > y),$$

and its moment generating function $M_Z(t) = Ee^{tZ}$ is given in the following

**Proposition 1.** *For an exp($\mu, \rho$) random variable $Z$ we have*

$$M_Z(t) = \frac{(1-\rho)(\mu - t)}{\mu(1-\rho) - t}.$$

*Proof.* Splitting the expected value into the discrete and the continuous parts we obtain

$$Ee^{tZ} = (1-\rho) + \rho \int_0^\infty e^{tx} \mu(1-\rho) e^{-\mu(1-\rho)x} dx.$$

We recognize the integral as the moment generating function of an $\exp(\mu(1-\rho))$ variable, and thus the above equation can be rewritten as

$$(1-\rho) + \frac{\rho\mu(1-\rho)}{\mu(1-\rho) - t} = \frac{(1-\rho)(\mu - t)}{\mu(1-\rho) - t}.$$

□

Proposition 1 yields immediately the moments of an $\exp(\mu, \rho)$ random variable $Z$ as $EZ^n = \rho n![\mu(1-\rho)]^{-n}$. The following result is the key to our more general claim:

**Proposition 2.** *If $Z$ is $\exp(\mu, \rho)$ and $W$ is $\exp(\mu)$, with $Z$ and $W$ independent, then $X = Z + W$ is $\exp(\mu(1-\rho))$.*

*Proof.* Elementary calculations with the moment generating functions allow us to write:

$$M_X(t) = M_Z(t)M_W(t) = \frac{(1-\rho)(\mu - t)}{\mu(1-\rho) - t}\frac{\mu}{\mu - t} = \frac{\mu(1-\rho)}{\mu(1-\rho) - t},$$

and we recognize in this expression the moment generating function of an $\exp(\mu(1-\rho))$ random variable, so we are done. □

Now we are ready to state our main result:

**Proposition 3.** *If $X$ is $\exp(\alpha)$ then for every $n \geq 0$ we can write*

$$X = X_1 + \cdots + X_n + X_{n+1},$$

*where $X_i$ is $\exp(\frac{\alpha}{(1-\rho)^i}, \rho)$ for $1 \leq i \leq n$, $X_{n+1}$ is $\exp(\frac{\alpha}{(1-\rho)^n})$, and all $X_i, 1 \leq i \leq n+1$ are independent.*

*Proof.* The simplest way to go about seems to be induction, the case $n = 1$ being proposition 2. Alternatively, multiplying all the expressions for the generating functions gets the results after a lot of cancellations. □

**Final comments.** Allen (1978) [1] shows laboriously, working directly with the interrarival times of an M/M/1 queue, that the waiting time in the queue and the total time in the system (waiting time plus service time) are $\exp(\mu, \rho)$ and $\exp(\mu(1-\rho))$, respectively, where $\mu$ is the exponential service rate and $\rho = \frac{\lambda}{\mu}$, where $\lambda$ is the interarrival exponential rate. This fact, plus the fact that the waiting and service times are independent show a particular case of our proposition 2. Our proof is based on generating functions and does not depend on any underlying queuing process.

An anonymous reviewer pointed out that an $\exp(\mu, \rho)$ variable can be thought of as the product of a Bernoulli variable with parameter $\rho$ and an independent $\exp(\mu(1-\rho))$ variable and thus all the results could be given in terms of Bernoulli and exponential variables, avoiding the use of $\exp(\mu, \rho)$ variables. We choose to keep them as a reminder of the M/M/1 queue that inspired this note: the probability mass at 0 of the $\exp(\mu, \rho)$ variable accounts for an arriving customer finding no customers in the system; if the arriving customer, however, finds $n$ customers already in the system, it must wait a gamma$(n, \mu)$ time; it is remarkable that after conditioning on the number of customers $n$, the distribution of the waiting time is a plain $\exp(\mu, \rho)$.

## References

[1] Allen A.O. *Probability, Statistics and Queuing Theory.* Academic Press, Orlando, 1978.

[2] Lamperti J. *Probability: a Survey of the Mathematical Theory.* W. A. Benjamin, New York, 1966.